

# ULTRAMETRIC AND TREE POTENTIAL

Claude DELLACHERIE <sup>\*</sup>, Servet MARTINEZ <sup>†</sup>, Jaime SAN MARTIN <sup>‡</sup>

February 1, 2008

## Abstract

We study infinite tree and ultrametric matrices, and their action on the boundary of the tree. For each tree matrix we show the existence of a symmetric random walk associated to it and we study its Green potential. We provide a representation theorem for harmonic functions that includes simple expressions for any increasing harmonic function and the Martin kernel. In the boundary, we construct the Markov kernel whose Green function is the extension of the matrix and we simulate it by using a cascade of killing independent exponential random variables and conditionally independent uniform variables. For ultrametric matrices we supply probabilistic conditions to study its potential properties when immersed in its minimal tree matrix extension.

## 1 Introduction and Basic Notation

### 1.1 Introduction

Here we study ultrametric and tree matrices, the random walk they induce on trees and its potential theory. There exists a broad literature in this field (a complete state-of-the-art study can be found in [23]). The main difference between our work and most part of this literature, is that our starting point is not a random walk on a tree, but a tree matrix or more general, an ultrametric matrix. In this viewpoint, the random walk is constructed

---

<sup>\*</sup>Laboratoire Raphaël Salem, UMR 6085, Université de Rouen, Site Colbert, 76821 Mont Saint Aignan Cedex, France; Email: Claude.Dellacherie@univ-rouen.fr. This author thanks support from Nucleus Millennium P04-069-F for his visit to CMM-DIM at Santiago

<sup>†</sup>CMM-DIM; Universidad de Chile; Casilla 170-3 Correo 3 Santiago; Chile; Email: smartine@dim.uchile.cl. The author's research is supported by Nucleus Millennium Information and Randomness P04-069-F.

<sup>‡</sup>CMM-DIM; Universidad de Chile; Casilla 170-3 Correo 3 Santiago; Chile; Email: jsanmart@dim.uchile.cl. The author's research is supported by Nucleus Millennium Information and Randomness P04-069-F.

from the matrix, a nontrivial fact, even in the finite case. Hence, most of the concepts must be expressed with respect to the matrix, that turns to have two representations. One in the tree as the sum of a potential and a harmonic basis. The other one in the boundary of the tree as the potential of a Markov process. Our results are not a simple translation of well-known results from walks on trees to matrices. New phenomenon appear: the formula for monotone harmonic functions; the predictable representation property of tree matrices, that is the keystone for a wide class of relations including the Martin kernel at  $\infty$ ; the formula relating different levels of the process in the boundary which allow us to simulate it, in a constructive way. Below we give the framework of our work and summarize some of the main results.

An ultrametric matrix  $U = (U_{ij} : i, j \in I)$  is a symmetric nonnegative matrix verifying the ultrametric inequality  $U_{ij} \geq \min\{U_{ik}, U_{kj}\}$  for all  $i, j, k \in I$ . When  $I$  is finite it was shown in [31], [15], that the inverse  $U^{-1}$  of a nonsingular ultrametric matrix  $U$  is a diagonal dominant Stieltjes matrix (see [35] for a linear algebra proof of this fact). Then,  $U$  is proportional to the Green potential of a subMarkov kernel  $P$ , that is  $U = \alpha \sum_{n \geq 0} P^n$ . Thus, if we consider for  $i \neq j$  the distance  $d(i, j) = 1/U_{ij}$ , then  $d$  is an ultrametric distance and  $1/d$  is a Green potential (a phenomenon that happens in  $\mathbb{R}^3$  with the Newtonian potential and the Euclidian distance, or in  $\mathbb{R}^d$ ,  $d \geq 3$ , when we allow an increasing function of the Euclidian distance).

Tree matrices are a special case of ultrametric matrices. They are defined by a rooted tree  $(I, \mathcal{T})$  (with root  $r$ ) and a strictly increasing function  $w : \{|k| : k \in I\} \rightarrow \mathbb{R}_+$ , where  $|k|$ , the level of  $k$ , is the length of the geodesic from a site  $k$  to  $r$ . Then, the tree matrix  $U$  is defined as  $U_{ij} = w_{|i \wedge j|}$ , with  $i \wedge j$  been the farthest vertex from  $r$  that is common to the geodesic from  $i$  and  $j$  to  $r$ . When  $I$  is finite,  $U$  is the potential of a Markov process, whose skeleton is a simple symmetric random walk on the tree, only defective at the root. We also mention here that every other ultrametric matrix is obtained by restriction of this class (see [15]). That is for every ultrametric matrix  $U$  there exists a minimal extension tree matrix  $\tilde{U}$ , defined on  $(\tilde{I}, \tilde{\mathcal{T}})$ , such that  $U = \tilde{U}|_I$ . This minimal tree  $\tilde{\mathcal{T}}$  has all the information that is required to understand the one step transitions of the Markov process associated to  $U$ . In fact,  $P_{ij} > 0$  if and only if the geodesic in  $\tilde{\mathcal{T}}$  joining  $i$  and  $j$  does not contain other points in  $I$ .

One of the purposes of this paper is to extend this study to countably infinite ultrametric and tree matrices. Each ultrametric matrix  $U$  defines a natural kernel  $W$  in the boundary  $\partial_\infty$  of the tree. This class of operators were already considered in [28] and [29], where a deep study of potential properties is done, mainly in connection to dimension and capacity on the boundary.

We show  $W$  is a stochastic integral operator whose associated filtration  $\mathcal{F} = (\mathcal{F}_k)$  is given by the tree structure, see Proposition 3.3. The operator  $W$  allows to represent harmonic functions in the infinite tree (see Corollary 3.1). This representation is an alternative to the well known Martin kernel representation, supplied for example in the basic reference [11] and in [39]. We describe the set of increasing (along the branches) harmonic functions as those functions that can be written in terms of  $U$ , see Theorem

3.1. Also, we characterize the set of bounded harmonic functions which are the difference of two harmonic increasing functions (see Theorem 3.2).

In the finite setting, a tree matrix  $U$  is the potential of a continuous time Markov chain, the leaves of the tree being reflecting states (see Proposition 2.2). Nevertheless, in the infinite transient case, each column of  $U$  is the sum of a potential and a nontrivial harmonic function, as follows from relation (3.2). This last result uses two main ingredients. The first one comes from the finite case analysis: when imposing Dirichlet boundary conditions at the boundary, a finite tree matrix is the sum of the potential matrix and a matrix whose columns generate the harmonic functions (see Proposition 2.4). The second element is the exit measure  $\mu$  at the boundary.

We mainly consider the potential of tree matrices for Markov semigroups defective at the root, because this is natural in the finite case. But, in the transient infinite case we can reflect the process at the root as we do in section 4 and by a limit procedure we can represent the Martin kernel in a similar way as for the absorbed chain, see Theorem 4.1. Also explicit computations for homogeneous trees are done, retrieving some known formulae ([12], [39]).

In section 5 we study ultrametric matrices  $U = (U_{ij} : i, j \in I)$ . Under some explicit hypotheses, we associate to  $U$  a minimal tree matrix  $\tilde{U} = (\tilde{U}_{\tilde{i}\tilde{j}} : \tilde{i}, \tilde{j} \in \tilde{I})$  extending it, with a natural immersion of the sites  $I$  into  $\tilde{I}$ . In Theorem 5.1 we show that a canonical generator  $Q$  can be associated to  $U$  with the help of the generator  $\tilde{Q}$  associated to  $\tilde{U}$ ; and in Theorem 5.2 it is shown that the harmonic functions defined by  $Q$  can be retrieved from the harmonic functions defined by  $\tilde{Q}$ . The key hypothesis is that a random walk starting from  $\tilde{I} \setminus I$  is trapped at the cemetery or it reaches  $I$  with probability one.

Let us turn to the process in the boundary of the tree. The fact that  $W$  is a stochastic integral operator reveals to be the main property which allow us to study the generator  $-W^{-1}$ . In Theorem 6.1 we describe the transition probability kernel of the subMarkov semigroup  $(e^{-tW^{-1}})$  acting on the boundary and having  $U$  as the kernel potential. In Theorem 6.2 we supply a recursive formula satisfied by the process in terms of: the killing time, the process killed at a successor of the root, and the process starting afresh from the distribution  $\mu$ . This allows us to give a constructive simulation of the process in terms of exponential random variables (killing times) and independent random variables distributed  $\mu$  conditioned to the atoms of the natural filtration.

There is a large literature on stochastic processes on the  $p$ -adic field. See for example the works of [1], [2], [3], [4], [25], [27] (see also the references therein). We notice that in these works there is a natural measure in the boundary, the Haar measure for the  $p$ -adic tree, or an absolutely continuous probability measure with respect to the Haar measure for the  $p$ -adic field. In our work the tree needs to be locally finite, but no other hypothesis is needed as homogeneity. Even with this generality, the exit measure at  $\infty$  fulfills the requirements allowing us to describe the process at the boundary. We also mention here, among others, the works of [26] and [19] in local fields and the work of [5] in disconnected spaces.

Ultrametricity is an important tool in applied areas: taxonomy (see [8]); the problem of maximal flow on finite graphs, namely the Theorem of Gomory-Hu (see [7]); statistical physics (see [18]) to explore the ultrametric Parisi solution to spin-glass models (see [34], [38] and references therein).

One of the tools we use in this work is the notion of stochastic integral operator (s.i.o.), which is the natural framework in which ultrametricity appears in stochastic analysis. An operator  $Y$  acting on a space  $L^2$  is an s.i.o. (see [17]) if for some filtration  $\mathcal{F} = (\mathcal{F}_t)$ ,  $Y$  can be written as

$$Yf = \int_0^\infty H_t d\mathbb{E}(f|\mathcal{F}_t) \text{ where } H = (H_t) \text{ is a } \mathcal{F} - \text{predictable process.}$$

The fact that  $H$  is predictable will play a fundamental role in the analysis of  $W$ . The characterization of s.i.o. on countable spaces led to study the relations between ultrametric matrices and filtrations (see [14]). On the other hand the continuous version of ultrametric matrices needs to consider operators of the form  $V = \int_0^\infty \mathbb{E}(|\mathcal{F}_t|) dG_t$ , where  $(G_t)$  is a bounded increasing and adapted process. In [16] it is shown that these operators are Markov potential kernels (a proof of it that uses backward stochastic differential equations can be found in [21]). This result is in the spirit and constitutes a generalization of the one obtained in [10].

## 1.2 Trees

Here we fix notation and recall some well-known notions on trees. Let  $(I, \mathcal{T})$  be a connected non-oriented and locally finite tree.  $I$  is the set of sites and  $\mathcal{T} \subset I \times I$  is the set of links. Two sites  $i, j$  are neighbors if  $(i, j) \in \mathcal{T}$ . The set of sites with a unique neighbor is called the extremal set and is denoted by  $\mathcal{E}$ . The geodesic joining  $i$  and  $j$  is denoted by  $geod(i, j)$  and its length is written  $|i - j|$ . In particular  $g(i, i)$  only contains  $i$  and its length is 0. We assume the tree is rooted by  $r \in I$  and we write  $|i| = |i - r|$ . We introduce the following order relation on  $I$ :

$$i \preceq j \text{ if } i \in geod(r, j). \quad (1.1)$$

The element  $i \wedge j = \max(geod(r, i) \cap geod(r, j))$  denotes the  $\preceq$  - minimum between  $i$  and  $j$ . For  $i \in I \setminus \{r\}$  there is a unique element  $i^-$  verifying  $(i^-, i) \in \mathcal{T}$  and  $i^- \preceq i$ , called the predecessor of  $i$ . It verifies  $|i^-| = |i| - 1$ . The set of successors of  $i \in I$  is denoted by  $S_i = \{j \in I : j^- = i\}$ , it is a finite set that could be empty. By  $i^+$  we mean a generic element of  $S_i$  and  $\mathcal{L} = \{i \in I : S_i = \emptyset\}$  is the set of leaves of the tree. We notice that  $\mathcal{L} \subseteq \mathcal{E}$ , and  $r$  is the only point that could be extremal without being a leaf. The branch of the tree born at  $i \in I$ , is denoted by  $[i, \infty) = \{j \in I : i \preceq j\}$ .

Assume that  $I$  is countably infinite. An infinite path  $(i_n \in I : n \in \mathbb{N})$  in the tree with origin  $i_0$ , is such that  $(i_n, i_{n+1}) \in \mathcal{T}$  for every  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ . If all the  $i_n$  are different this path is called an infinite chain. The following relation

$$(i_n : n \in \mathbb{N}) \sim (j_n : n \in \mathbb{N}) \Leftrightarrow |\{i_n : n \in \mathbb{N}\} \cap \{j_n : n \in \mathbb{N}\}| = \infty,$$

is an equivalence relation in the set of chains. The quotient set is the boundary of the tree  $(I, \mathcal{T})$  (see [11]) and we denote it by  $\partial_\infty$

For every  $i \in I$  and  $\xi \in \partial_\infty$  there exists a unique chain of origin  $i$  which is in the equivalence class  $\xi$ , and it is called the *geodesic* between  $i$  and  $\xi$ , and denoted by  $\text{geod}(i, \xi)$ . For a fixed  $\xi \in \partial_\infty$  and  $n \in \mathbb{N}$ , we denote by  $\xi(n)$  the unique point in the geodesic  $\text{geod}(r, \xi)$  such that  $|\xi(n)| = n$ , thus  $\xi(0) = r$ . Let  $(i_n : n \geq 0)$  be an infinite path, the following criterion stated in [11], is useful to establish convergence to a point in the boundary,

$$(\forall j \in I : |\{n \in \mathbb{N} : i_n = j\}| < \infty) \Rightarrow \exists! \xi = \lim_{n \rightarrow \infty} i_n \in \partial_\infty. \quad (1.2)$$

In this case there exists a subsequence  $(k_n : n \geq 0)$  verifying  $\text{geod}(i_0, \xi) = (i_{k_n} : n \geq 0)$ .

For  $i \in I$ ,  $\xi \in \partial_\infty$  we put  $i \preceq \xi$  if  $i \in \text{geod}(r, \xi)$ . Hence we can extend  $\wedge$  to  $I \cup \partial_\infty$  by

$$\xi \wedge \eta = \max(\text{geod}(r, \xi) \cap \text{geod}(r, \eta)). \quad (1.3)$$

Hence,  $\xi \wedge \xi = \xi$  and if  $\xi \neq \eta$  then  $\xi \wedge \eta \in I$ . In this last case  $\xi \wedge \eta = i$  if and only if  $\xi(|i|) = \eta(|i|)$  and  $\xi(n) \neq \eta(n)$  for  $n > |i|$ .

The extended subtree hanging from  $i \in I$  is  $[i, \infty] = \{z \in I \cup \partial_\infty : i \preceq z\}$ . The set  $I \cup \partial_\infty$  is endowed with the topology  $\mathbf{T}$  generated by the basis of open sets  $\mathcal{A} = \{[i, \infty] : i \in I\} \cup \{\{i\} : i \in I\}$ . The sets in  $\mathcal{A}$  are open and closed in  $\mathbf{T}$ . The topological space  $(I \cup \partial_\infty, \mathbf{T})$  is compact, totally discontinuous and metrically generated, the trace topology on  $I$  is the discrete one and  $I$  is an open dense subset in  $I \cup \partial_\infty$ . Also  $\mathcal{A}$  is a semi-algebra generating the Borel  $\sigma$ -algebra  $\sigma(\mathbf{T})$ . We use the following notation

$$\partial_\infty(i) = [i, \infty] \cap \partial_\infty = \{\eta \in \partial_\infty : i \preceq \eta\}. \quad (1.4)$$

The class of sets  $\mathcal{C} = \{\partial_\infty(i) : i \in I\}$  is a basis of open (and closed) sets generating  $\mathbf{T} \cap \partial_\infty$  and it is also a semi-algebra generating the trace of  $\sigma(\mathbf{T})$  on  $\partial_\infty$ . Therefore for  $\xi \in \partial_\infty$  :  $\xi = \lim_{n \rightarrow \infty} \xi(n)$  and  $\partial_\infty(\xi(n)) = \{\eta \in \partial_\infty : |\xi \wedge \eta| \geq n\}$ .

It will be useful to add an state  $\partial_r \notin I$  and the oriented link  $(r, \partial_r)$ . We put  $r^- = \partial_r$  and  $|\partial_r| = -1$ .

In the sequel we adopt the following notation. For any nonempty subset  $J \subseteq I$  we denote by  $\mathbb{I}_{J \times J}$  the identity  $J \times J$  matrix. If  $M$  is an  $I \times I$  matrix and  $J, K \subseteq I$  are nonempty, the matrix  $M_{JK} = (M_{jk} : j \in J, k \in K)$  (also denoted by  $M_{J,K}$ ) is the restriction of  $M$  to  $J \times K$ . By  $\mathbf{1}_A$  we mean the characteristic function of a set  $A$ , and  $\mathbf{1}$  is the constant function taking the value 1 in its domain of definition.

## 2 Tree Matrices

In [15] we have introduced the notion of tree matrices in the finite case. Here we give a general version of it. Let  $(I, \mathcal{T})$  be a tree with root  $r$ . Put  $\mathbf{N} = \{|i| : i \in I\}$ , which is equal to  $\mathbb{N}$  when the tree is infinite.

**Definition 2.1** A tree matrix  $U = (U_{ij} : i, j \in I)$  is defined by an strictly positive and strictly increasing function  $w : \mathbf{N} \rightarrow (0, \infty)$  as follows,

$$U_{ij} = w_{|i \wedge j|} \text{ for } i, j \in I.$$

The matrix  $U$  is strictly positive and symmetric, and it verifies  $U_{ij} = U_{i \wedge j, i \wedge j}$ . In particular  $U_{i^-i} = U_{i^-i^-} = w_{|i|-1}$  when  $i^- \in I$ . Notice that  $U_{i^+i^+} = w_{|i|+1}$  does not depend on the particular element  $i^+ \in S_i$ . We extend  $U$  to  $I \cup \{\partial_r\}$  by putting  $U_{i\partial_r} = U_{\partial_r i} = w_{-1} = 0$  for every  $i \in I \cup \{\partial_r\}$ .

By using (1.3) we can extend  $U$  to  $I \cup \partial_\infty$  in the following way

$$\text{for } \xi, \eta \in \partial_\infty, \quad U_{\xi\eta} = w_{|\xi \wedge \eta|} \text{ if } \xi \neq \eta \text{ and } U_{\xi\xi} = \lim_{n \rightarrow \infty} U_{\xi(n)\xi(n)}. \quad (2.1)$$

This extension is continuous in both variables:  $U_{\xi\eta} = \lim_{n \rightarrow \infty, m \rightarrow \infty} U_{\xi(n)\eta(m)}$  for  $\xi, \eta \in \partial_\infty$ .

We associate to  $U$  a symmetric matrix  $Q = (Q_{ij} : i, j \in I)$  supported by the tree and the diagonal, that is  $Q_{ij} = 0$  if  $i \neq j$  and  $(i, j) \notin \mathcal{T}$ . This matrix  $Q$  is given by

$$\begin{aligned} Q_{ii^-} &= Q_{i^-i} = (w_{|i|} - w_{|i|-1})^{-1} \text{ for } i^-, i \in I; \\ Q_{ii} &= -((w_{|i|} - w_{|i|-1})^{-1} + |S_i|(w_{|i|+1} - w_{|i|})^{-1}) \text{ for } i \in I. \end{aligned} \quad (2.2)$$

Observe that  $Q_{ii^+} = Q_{i^+i} = (w_{|i|+1} - w_{|i|})^{-1}$  does not depend on  $i^+ \in S_i$ . When  $i \in \mathcal{L}$  is a leaf, then  $Q_{ii} = -Q_{ii^-}$ . The matrix  $Q$  verifies  $Q_{ij} \geq 0$  if  $j \neq i$  and  $\sum_{j \in I} Q_{ij} \leq 0$  for  $i \in I$ . Then  $Q$  is a  $q$ -matrix, it is conservative in the sites  $i \in I \setminus \{r\}$ , that is  $\sum_{j \in I} Q_{ij} = 0$ ,

and defective at  $r$  since  $\sum_{j \in I} Q_{rj} = -w_0^{-1}$ . We call  $\widehat{Q}$  the extension of  $Q$  to  $I \cup \{\partial_r\}$ , given by

$$\widehat{Q}_{r\partial_r} = w_0^{-1} \text{ and } \widehat{Q}_{i\partial_r} = 0 \text{ for } i \neq r, i \in I \cup \{\partial_r\}. \quad (2.3)$$

This extension is a nonsymmetric conservative  $q$ -matrix in  $I \cup \{\partial_r\}$ , having  $\partial_r$  as an absorbing state.

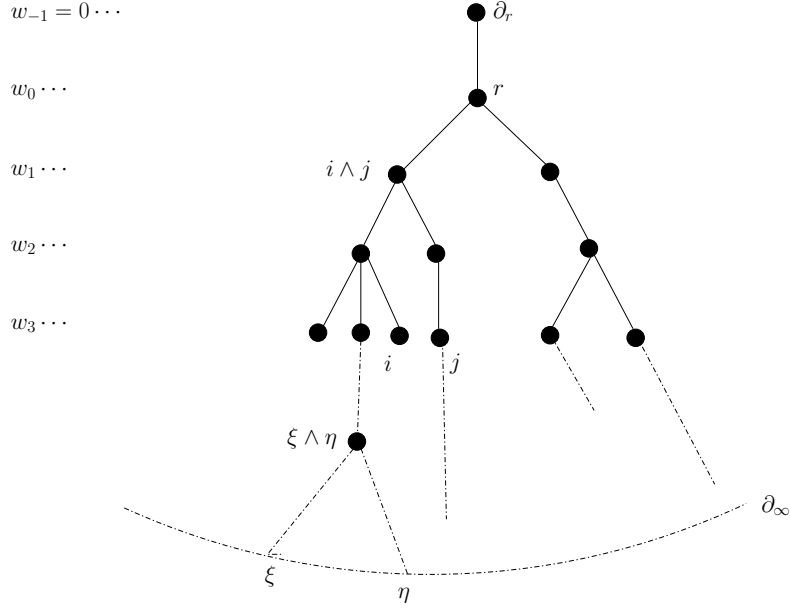


Figure 1: Tree Matrix

Observe that if  $M$  is an  $I \times I$  matrix then the formal products of matrices  $QM$  and  $MQ$  are well defined because each line and column of  $Q$  has finite support.

**Proposition 2.1** *The  $q$ -matrix  $Q$  verifies  $(-Q)U = U(-Q) = \mathbb{I}$ .*

**Proof.** From symmetry it suffices to show  $(-Q)U = \mathbb{I}$ . For  $i, k \in I$  we have

$$(QU)_{ik} = Q_{ii-}U_{i-k} + Q_{ii}U_{ik} + Q_{ii+} \sum_{j \in S_i} U_{jk}.$$

If  $k \wedge i \preceq i^-$  we have  $i \neq r$  and  $k \wedge i = k \wedge i^- = k \wedge i^+$ . Then  $(QU)_{ik} = 0$  because  $Q$  is conservative at  $i \in I$ .

For  $k = i$  we have

$$\begin{aligned} (QU)_{ii} &= Q_{ii-}U_{i-i} + Q_{ii}U_{ii} + |S_i|Q_{ii+}U_{ii} \\ &= Q_{ii-}U_{i-i} - Q_{ii-}U_{ii} - |S_i|Q_{ii+}U_{ii} + |S_i|Q_{ii+}U_{ii} = -Q_{ii-}(U_{ii} - U_{i-i}) = -1. \end{aligned}$$

The last case left to analyze is when  $k \wedge i^+ = i^+$  for some and a unique  $i^+ \in S_i$ . Then  $k \wedge i^- = i^-$ ,  $k \wedge i = i = k \wedge j$  for  $j \in S_i \setminus \{i^+\}$ . Hence

$$\begin{aligned} (QU)_{ik} &= Q_{ii-}U_{i-i^-} + Q_{ii}U_{ii} + (|S_i| - 1)Q_{ii+}U_{ii} + Q_{ii+}U_{i^+i^+} \\ &= (QU)_{ii} + Q_{ii+}(U_{i^+i^+} - U_{ii}) = -1 + 1. \end{aligned}$$

□

**Remark 2.1** As we shall see  $Q$  is a generator of a Markov process with state space  $I \cup \{\partial_r\}$ . Its discrete skeleton has transition probabilities given by

$$p_{ij} = \frac{Q_{ij}}{\sum_{k \in S_i \cup \{i^-\}} Q_{ik}} \text{ for } j \in S_i \cup \{i^-\}.$$

In the electrical circuits interpretation, this corresponds to a chain whose conductances are given by  $C_{ii^-} := Q_{ii^-}$  (see [24] section 9, and [29] section 2).

Let us study more closely the case when  $(I, \mathcal{T})$  is a finite tree rooted at  $r$ . Since the state space is finite, the matrix  $Q = -U^{-1}$  is an infinitesimal generator defective only at  $r$ . Let  $(X_t : 0 \leq t < \zeta)$  be the associated Markov process taking values on  $I$ , with lifetime  $\zeta$ . We denote  $(\hat{X}_t : 0 \leq t < \infty)$  the Markov chain associated to the extension  $\hat{Q}$ . We notice that  $\partial_r$  is an absorbing state for  $\hat{X}$ . Let  $T_{\partial_r} = \inf\{t \geq 0 : \hat{X}_t = \partial_r\}$ . Then, when starting from an state in  $I$ , the chains  $(\hat{X}_t : 0 \leq t < T_{\partial_r})$  and  $(X_t : 0 \leq t < \zeta)$ , have the same distribution. Hence  $\zeta = T_{\partial_r}$ . Therefore, if necessary we can assume that  $X$  is defined in  $I \cup \{\partial_r\}$ .

**Proposition 2.2** Let  $(I, \mathcal{T})$  be a finite tree rooted at  $r$ . Then  $U$  is the potential matrix of the chain  $(X_t : 0 \leq t < \zeta)$ , that is  $U = \int_0^\infty e^{tQ} dt$  or equivalently  $U_{ij} = \mathbb{E}_i\left(\int_0^\infty \mathbf{1}_{\{X_t=j\}} dt\right)$ .

**Proof.** Since  $(e^{tQ})$  is the semigroup of  $(X_t : 0 \leq t < \zeta)$  we get  $U = -Q^{-1} = \int_0^\infty e^{tQ} dt$ .  $\square$

We set  $n+1 = |\mathbf{N}| = \max\{|i| : i \in I\}$ . Consider the sets

$$B^{n+1} = \{i \in I : |i| = n+1\} \text{ and } \tilde{B}^n = \{i \in I : |i| = n, S_i \neq \phi\}.$$

Hence  $B^{n+1} = \cup_{i \in \tilde{B}^n} S_i$ . To avoid the trivial situation we assume  $n \geq 1$ . We will also set  $I^m = \{i \in I : |i| \leq m\}$ , so  $I = I^{n+1}$ .

We denote by  $T_i = \inf\{t \geq 0 : X_t = i\}$  the hitting time of  $i \in I$ , and by  $T_{\tilde{B}^n} := \inf\{T_i : i \in \tilde{B}^n\}$  and  $T_{B^{n+1}} := \inf\{T_i : i \in B^{n+1}\}$  the hitting times of  $\tilde{B}^n$  and  $B^{n+1}$ , respectively, .

Let  $Q_{I^n I^n}$  be the restriction of  $Q$  to  $I^n \times I^n$ . The chain  $(X_t : t < T_{\partial_r} \wedge T_{B^{n+1}})$  killed at  $B^{n+1} \cup \{\partial_r\}$  has generator  $Q_{I^n I^n}$  and semigroup  $(e^{tQ_{I^n I^n}})$ . Its potential  $V^{(n)} := -(Q_{I^n I^n})^{-1}$  verifies

$$V_{ij}^{(n)} = \mathbb{E}_i\left(\int_0^{T_{\partial_r} \wedge T_{B^{n+1}}} \mathbf{1}_{\{X_t=j\}} dt\right) \text{ for } i, j \in I^n.$$

Further consider the  $q$ -matrix  $\bar{Q}^{(n)}$  defined in  $I_n$  by

$$\bar{Q}_{I^n \setminus \tilde{B}^n, I^n}^{(n)} = Q_{I^n \setminus \tilde{B}^n, I^n} \text{ and } \bar{Q}_{\tilde{B}^n I^n}^{(n)} = 0.$$

**Definition 2.2** Given a  $q$ -matrix  $Q$  on the set  $I$ , we say that a function  $h : I \rightarrow \mathbb{R}$  is  $Q$ -harmonic if it verifies  $Qh = 0$ .



From the definition it is clear that  $h$  is  $Q$ -harmonic iff  $e^{tQ}h = h$ , for all  $t \geq 0$ . In the next proposition we present a result that we will need in what follows. Its proof is standard and it is based on the Doob' sampling theorem.

**Proposition 2.3** *A function  $h : I^n \rightarrow \mathbb{R}$  is  $\bar{Q}^{(n)}$ -harmonic if and only if*

$$\mathbb{E}_i(h(X_{\tau \wedge T_{\tilde{B}^n}})) = h(i) \text{ for } i \in I^n \text{ and any stopping time } \tau \leq \infty.$$

The class of  $\bar{Q}^{(n)}$ -harmonic functions, denoted by  $\mathcal{H}^n$ , is a linear space with dimension  $\dim \mathcal{H}^n = |\tilde{B}^n|$ . Indeed, for each  $k \in \tilde{B}^n$  the function  $h^k(i) = \mathbb{E}_i(\mathbf{1}_k(X_{T_{\tilde{B}^n}}))$  is the unique harmonic function which verifies  $h^k(j) = \delta_{kj}$  for  $j \in \tilde{B}^n$ . The class of these harmonic functions constitutes a basis for  $\mathcal{H}^n$ .

**Proposition 2.4** *Let  $(I, \mathcal{T})$  be a finite tree rooted at  $r$ . The matrix  $H := U_{I^n I^n} - V^{(n)}$  is symmetric, and its columns generate the space  $\mathcal{H}^n$  of  $\bar{Q}^{(n)}$ -harmonic functions. Moreover, the columns of  $U_{I^n \tilde{B}^n}$  is a basis of this space.*

**Proof.** First, let us introduce the matrices  $W = (W_{ik} : i \in I^n, k \in \tilde{B}^n)$ ,  $E = (E_{i\ell} : i \in I^n, \ell \in B^{n+1})$ ,  $D = (D_{ik} : i \in I^n, k \in \tilde{B}^n)$ , whose terms are

$$W_{ik} = \mathbb{P}_i\{X_{T_{\tilde{B}^n}} = k\}, \quad E_{i\ell} = \mathbb{P}_i\{X_{T_{B^{n+1}}} = \ell\}, \quad D_{ik} = \mathbb{P}_i\{X_{T_{B^{n+1}}} \in S_k\}.$$

Let  $W^k$  be the  $k$  column of  $W$ , with  $k \in \tilde{B}^n$ . We notice that  $h^k = W^k$  then  $(W^k : k \in \tilde{B}^n)$  is a basis of  $\mathcal{H}^n$ . In particular  $\bar{Q}^{(n)}W^k = 0$ .

From definition  $D_{ik} = \sum_{\ell \in S_k} E_{i\ell}$ , or equivalently  $D = EM^t$  where  $M^t$  is the transposed of the incidence matrix  $M = (M_{k\ell} : k \in \tilde{B}^n, \ell \in B^{n+1})$ , with  $M_{k\ell} = 1$  if  $\ell \in S_k$  and  $M_{k\ell} = 0$  otherwise.

Let  $i \in I^n$  and  $k \in \tilde{B}^n$ . Since

$$\mathbb{P}_i\{T_k < \infty\} = \sum_{j \in \tilde{B}^n} \mathbb{P}_i\{X_{T_{\tilde{B}^n}} = j\} \mathbb{P}_j\{T_k < \infty\} \text{ and } U_{ik} = \mathbb{P}_i\{T_k < \infty\} U_{kk},$$

we find  $U_{ik} = \sum_{j \in \tilde{B}^n} \mathbb{P}_i\{X_{T_{\tilde{B}^n}} = j\} U_{jk}$ . Hence we obtain

$$U_{I^n \tilde{B}^n} = W U_{\tilde{B}^n \tilde{B}^n} \text{ and so } W = U_{I^n \tilde{B}^n} (U_{\tilde{B}^n \tilde{B}^n})^{-1}. \quad (2.4)$$

Analogously we get  $E = U_{I^n B^{n+1}} (U_{B^{n+1} B^{n+1}})^{-1}$ . From the equality  $D = EM^t$  we find  $D = U_{I^n B^{n+1}} (U_{B^{n+1} B^{n+1}})^{-1} M^t$ . Since  $U_{i\ell} = w_{|i|} = U_{ik}$  when  $k \in \tilde{B}^n$ ,  $\ell \in S_k$ , we obtain

$$U_{I^n B^{n+1}} = U_{I^n \tilde{B}^n} M, \quad (2.5)$$

and then  $D = U_{I^n \tilde{B}^n} M (U_{B^{n+1} B^{n+1}})^{-1} M^t$ . Let us show

$$H = U_{I^n \tilde{B}^n} M (U_{B^{n+1} B^{n+1}})^{-1} M^t U_{\tilde{B}^n I^n}, \quad (2.6)$$

or equivalently  $H = DU_{\tilde{B}^n I^n}$ . For  $i, j \in I^n$  we have

$$\begin{aligned} U_{ij} &= \mathbb{E}_i \left( \int_0^\infty \mathbf{1}_{\{X_t=j\}} dt \right) \\ &= \mathbb{E}_i \left( \int_0^{T_{B^{n+1}}} \mathbf{1}_{\{X_t=j\}} dt \right) + \mathbb{E}_i \left( T_{B^{n+1}} < \infty, \mathbb{E}_{X_{T_{B^{n+1}}}} \left( \int_0^\infty \mathbf{1}_{\{X_t=j\}} dt \right) \right). \end{aligned}$$

Hence  $U_{ij} = V_{ij}^{(n)} + \sum_{\ell \in B^{n+1}} \mathbb{P}_i \{X_{T_{B^{n+1}}} = \ell\} U_{\ell j}$ , or equivalently

$$U_{ij} = V_{ij}^{(n)} + \mathbb{E}_i(T_{B^{n+1}} < \infty, U_{X_{T_{B^{n+1}}}} j). \quad (2.7)$$

Then  $H_{ij} = \mathbb{E}_i(T_{B^{n+1}} < \infty, U_{X_{T_{B^{n+1}}}} j)$  and by using (2.5) we find

$$H_{ij} = \sum_{\ell \in B^{n+1}} \mathbb{P}_i \{X_{T_{B^{n+1}}} = \ell\} U_{\ell j} = \sum_{k \in \tilde{B}^n} \mathbb{P}_i \{X_{T_{B^{n+1}}} \in S_k\} U_{kj} \text{ for } i, j \in I^n,$$

which gives us  $H = DU_{\tilde{B}^n I^n}$ , that is (2.6) holds. From (2.6) we deduce  $\text{rank } H = \text{rank } U_{\tilde{B}^n I^n} = |\tilde{B}^n| = \dim \mathcal{H}^n$ . On the other hand, from (2.4) and (2.6) we get

$$H = WU_{\tilde{B}^n \tilde{B}^n} M(U_{B^{n+1} B^{n+1}})^{-1} M^t U_{\tilde{B}^n \tilde{B}^n} W^t. \quad (2.8)$$

From  $\bar{Q}^{(n)}W = 0$  we obtain  $\bar{Q}^{(n)}H = 0$ . Therefore, the columns of  $H$  belong to the space  $\mathcal{H}^n$ . Given that  $\text{rank}(H) = \dim(\mathcal{H}^n)$  the columns of  $H$  generate this space. On the other hand from (2.6) the columns of  $U_{I^n \tilde{B}^n}$  generate  $\mathcal{H}^n$ . Since the rank of this matrix is equal the dimension of  $\mathcal{H}^n$  the Proposition is shown.  $\square$

### 3 Harmonic Functions and the Martin Kernel

From now on we assume that  $(I, \mathcal{T})$  is an infinite rooted tree. We also assume that each branch is infinite. We consider the minimal transition semigroup  $\hat{P}_t$  associated to  $\hat{Q}$  the extension of  $Q$  to  $I \cup \{\partial_r\}$  made in (2.3). One way to construct this semigroup is by truncating the state space by an increasing sequence of finite sets and then use [6] Proposition 2.14. Let  $\hat{X} = (\hat{X}_t : 0 \leq t < \hat{\zeta})$  be a time continuous Markov process with infinitesimal generator  $\hat{Q}$  and lifetime  $\hat{\zeta}$ . If we stop  $\hat{X}$  at the hitting time of  $\partial_r$  we obtain a Markov process  $X = (\hat{X}_t : 0 \leq t < \zeta)$  whose state space is  $I$  and lifetime  $\zeta = T_{\partial_r} \wedge \hat{\zeta}$ . The infinitesimal generator for  $X$  is given by  $Q$ . We denote by  $(P_t)$  the semigroup associated to  $X$  and by  $V = \int_0^\infty P_t dt$ , the potential induced on  $I$ . We will denote by  $Y = (Y_n : n \in \mathbb{N})$  the discrete skeleton on  $I$  induced by  $X$ .

Let  $I^n = \{i \in I : |i| \leq n\}$ . As in the previous section  $V^{(n)}$  is the potential associated to  $Q_{I^n I^n}$  and  $\mathcal{H}^n$  is the set of  $\bar{Q}^{(n)}$ -harmonic functions in  $I^n$ . Consider the chain  $X^{(n)} := (X_t : t < T_{\partial_r} \wedge T_{B^{n+1}})$  killed at  $B^{n+1} \cup \{\partial_r\}$ , with generator  $Q_{I^n I^n}$ . The Markov semigroup

is  $P_t^{(n)} = e^{tQ_{I^n I^n}}$  and  $V^{(n)} = \int_0^\infty P_t^{(n)} dt = -Q_{I^n I^n}^{-1}$  is the associated potential. Clearly we have  $(P_t^{(n)})_{ij} \leq (P_t^{(n+1)})_{ij}$  and  $V_{ij}^{(n)} \leq V_{ij}^{(n+1)}$  for  $i, j \in I^n$ . Moreover, by the Monotone Convergence Theorem their limits are  $(P_t)$  and  $V$ , respectively. From (2.7) we get  $V_{ij}^{(n)} \leq U_{ij}$ , then  $V \leq U$ .

Let us see, by a classical procedure (for instance see [11]), that  $X_\zeta$  is a well defined variable in  $I \cup \partial_\infty \cup \partial_r$ . In the case  $T_{\partial_r} < \infty$  this is obvious because  $T_{\partial_r} = \zeta$  and  $X_\zeta = \partial_r$ . So we can assume  $T_{\partial_r} = \infty$ . We define  $R_n = \inf\{t \geq 0 : |X_t| \geq n\}$  and  $R_\infty := \lim_{n \rightarrow \infty} R_n$ . An argument based on Borel Cantelli Lemma shows that the set of trajectories visiting a site  $j \in I$  an infinite number times by  $(Y_n)$ , has  $\mathbb{P}_i$ -measure 0. In fact for such trajectories we necessarily have  $T_{\partial_r} < \infty$ . The trajectories that visit each site of  $I$  only a finite number of times and are not absorbed at  $\partial_r$  must converge to a point in the boundary  $\partial_\infty$  (see (1.2)). Therefore  $\zeta = T_{\partial_r} \wedge R_\infty$  and  $X_\zeta$  is well defined. It verifies

$$X_\zeta = \partial_r \text{ if } T_{\partial_r} < R_\infty \text{ and } X_\zeta = \lim_{n \rightarrow \infty} X_{R_n} = \lim_{n \rightarrow \infty} X_\zeta(n) \in \partial_\infty \text{ if } R_\infty \leq T_{\partial_r}. \quad (3.1)$$

Here, as already introduced,  $X_\zeta(n)$  is the point at level  $n$  in  $\text{geod}(r, X_\zeta)$ .

The tree matrix is said to be *transient* whenever  $\mathbb{P}_r\{T_{\partial_r} < \infty\} < 1$  or equivalently  $\mathbb{P}_r\{X_\zeta \in \partial_\infty\} > 0$ . Otherwise, the tree matrix is said to be *recurrent*. This classification corresponds to the recurrence or transient property for the chain reflected at  $r$ . For a simple criterion on transience see [30].

Since  $U_{i\partial_r} = U_{\partial_r i} = 0$  for every  $i \in I \cup \{\partial_r\}$ , equality (2.7) can be written as

$$U_{ij} = V_{ij}^{(n)} + \mathbb{E}_i(U_{X_{T_{B^{n+1}}}} j).$$

From  $U_{X_{T_{B^{n+1}}}} j \leq U_{jj}$  and  $\lim_{n \rightarrow \infty} U_{X_{T_{B^{n+1}}}} j = U_{X_\zeta j}$   $\mathbb{P}_i$ -a.e., we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}_i(U_{X_{T_{B^{n+1}}}} j) = \mathbb{E}_i(U_{X_\zeta j}).$$

By combining these relations with  $\lim_{n \rightarrow \infty} V_{ij} = V_{ij}^{(n)}$ , allow us to get

$$U_{ij} = V_{ij} + \mathbb{E}_i(U_{X_\zeta j}) = V_{ij} + \int_{\partial_\infty} U_{\eta j} \mathbb{P}_i\{X_\zeta \in d\eta\}. \quad (3.2)$$

Given that  $V_{ij} = V_{ji} = \mathbb{P}_j\{T_i < \infty\}V_{ii}$  the following limit exists

$$V_{i\xi} := \lim_{j \rightarrow \xi} V_{ij} = V_{ii} \cdot \lim_{j \rightarrow \xi} \mathbb{P}_j\{T_i < \infty\} \geq 0, \text{ for } i \in I, \xi \in \partial_\infty. \quad (3.3)$$

Therefore, passing to the limit  $j \rightarrow \xi \in \partial_\infty$  in relation (3.2) and using the Monotone Convergence Theorem lead to

$$U_{i\xi} = V_{i\xi} + \int_{\partial_\infty} U_{\eta\xi} \mathbb{P}_i\{X_\zeta \in d\eta\}. \quad (3.4)$$

A conclusion derived from (3.2) is that the recurrent case  $\mathbb{P}_r\{X_\zeta \in \partial_\infty\} = 0$  is completely characterized by the equality  $V = U$ . In particular the tree matrix  $U$  is the potential of  $(X_t)$ .

In the transient case we denote by  $\mu$  the exit measure on the boundary  $\partial_\infty$ , that is the probability measure defined on  $\partial_\infty$  by

$$\mu(\bullet) = \mathbb{P}_r\{X_\zeta \in \bullet \mid X_\zeta \in \partial_\infty\}. \quad (3.5)$$

**Remark 3.1** *If  $U$  is unbounded, that is  $w_n$  tends to infinity as  $n$  increases, the measure  $\mu$  is atomless. In fact, from (3.4) we get*

$$\infty > w_0 = U_{r\xi} \geq \int_{\partial_\infty \setminus \{\xi\}} U_{\eta\xi} \mathbb{P}_r\{X_\zeta \in d\eta\} + \infty \cdot \mathbb{P}_r\{X_\zeta = \xi\}.$$

In what follows we concentrate on the transient case. Nevertheless, when appropriate, we shall point out the corresponding results for the recurrent case.

### 3.1 Harmonic Functions

In this subsection we study basic properties of the harmonic functions on  $I$ . We notice that the restriction of a  $\widehat{Q}$ -harmonic function to  $I$  is not necessarily  $Q$ -harmonic. An example of this is the constant **1** function. In fact, the unique  $\widehat{Q}$ -harmonic functions whose restrictions are  $Q$ -harmonics are those vanishing at  $\partial_r$ . Obviously the reciprocal also holds, that is, the only  $\widehat{Q}$ -harmonic extension of a  $Q$ -harmonic function is the one extended by 0 at  $\partial_r$ . In the sequel a harmonic function is to be understood as a  $Q$ -harmonic function, and for a function defined on a subset of  $I \cup \partial_\infty$  we assume implicitly that it takes the value 0 at  $\partial_r$ , unless otherwise is specified.

In what follows an important role is played by the function

$$\bar{g}(j) = \mathbb{P}_j\{T_{\partial_r} < \infty\}, \quad j \in I \cup \{\partial_r\}, \quad (3.6)$$

which is the Martin kernel for  $\widehat{Q}$  at  $\partial_r$ . We point out that both  $\bar{g}$  and  $1 - \bar{g}$  are  $\widehat{Q}$ -harmonics, but only  $1 - \bar{g}$  is  $Q$ -harmonic. We also note that  $\bar{g}$  is nonnegative and decreasing on each branch, which allows to define for  $\eta \in \partial_\infty$

$$\bar{g}(\eta) := \lim_{j \rightarrow \eta} \mathbb{P}_j\{T_{\partial_r} < \infty\}.$$

Given  $g : I \rightarrow \overline{\mathbb{R}}$  an extended real function defined on the tree, we consider the sequence of functions  $(g_n)$  defined on the boundary by

$$g_n(\xi) = g(\xi(n)) \text{ for } n \in \mathbb{N} \text{ and } \xi \in \partial_\infty.$$

This notion enable us to study limiting properties on the boundary for functions defined on the extended tree.

**Definition 3.1** Let  $g : I \rightarrow \overline{\mathbb{R}}$  and  $\varphi : \partial_\infty \rightarrow \overline{\mathbb{R}}$ . We put  $\lim g = \varphi$  pointwise (respectively  $\mu$ -a.e.) if  $\lim_{n \rightarrow \infty} g_n = \varphi$  pointwise (respectively  $\mu$ -a.e.).

Let  $\bar{R}_n := \inf\{t \geq 0 : |X_t| \geq n \text{ or } X_t = \partial_r\}$ . A standard argument gives,

$$h : I \rightarrow \mathbb{R} \text{ is harmonic} \Leftrightarrow [\forall n \geq 1, \forall \tau \text{ stopping time} : \forall i \in I, h(i) = \mathbb{E}_i(h(X_{\tau \wedge \bar{R}_n}))].$$

In the transient case, an application of the Dominated Convergence Theorem and the Fatou's Theorem gives that for any bounded harmonic function  $h : I \rightarrow \mathbb{R}$  the limit  $\varphi = \lim h$  exists  $\mu$ -a.e. and moreover

$$h(i) = \mathbb{E}_i(\varphi(X_\zeta)).$$

Indeed, this is a consequence of Theorem 2.6 in [11], because  $h$  is bounded if and only if  $h/(1 - \bar{g})$  is bounded. Thus, if  $h_1, h_2$  are bounded harmonic functions such that  $\lim h_1 = \lim h_2$   $\mu$ -a.e. then  $h_1 \equiv h_2$ . Obviously in the recurrent case the unique bounded harmonic function is  $h \equiv 0$ .

**Proposition 3.1** If  $U$  is bounded then the tree matrix is transient.

**Proof.** The function  $h(i) = U_{i\eta}$  is harmonic, bounded and non-zero which implies that the tree matrix must be transient.  $\square$

A distinguished class of harmonic functions is given by the Martin kernel at  $\infty$ , see [11], [24] or [37].

**Definition 3.2** The Martin kernel (at  $\infty$ ),  $\kappa : I \times \partial_\infty \rightarrow \mathbb{R}$  is given by

$$\kappa(i, \eta) := \lim_{j \rightarrow \eta} \frac{V_{ij}}{V_{rj}}, \text{ for } i \in I, \eta \in \partial_\infty.$$

It is well known that  $\kappa(\bullet, \eta)$  is a well defined harmonic function on  $I$  (see [11] or [37]).

Consider  $i \in I$ ,  $\xi \in \partial_\infty$  and  $n > |i \wedge \xi|$ . Take  $j = \xi(n)$  and denote  $C^n = \partial_\infty(\xi(n))$ . The strong Markov property implies

$$\mathbb{P}_i\{X_\zeta \in C^n\} = \mathbb{P}_i\{T_j < \infty\} \mathbb{P}_j\{X_\zeta \in C^n\} = \frac{V_{ij}}{V_{jj}} \mathbb{P}_{\xi(n)}\{X_\zeta \in C^n\}.$$

On the other hand  $\mathbb{P}_i\{X_\zeta \in C^n\} = \mathbb{P}_i\{T_{i \wedge \xi} < \infty\} \mathbb{P}_{i \wedge \xi}\{X_\zeta \in C^n\}$ . Then

$$\frac{V_{ij}}{V_{rj}} = \frac{\mathbb{P}_i\{X_\zeta \in C^n\}}{\mathbb{P}_r\{X_\zeta \in C^n\}} = \frac{\mathbb{P}_i\{T_{i \wedge \xi} < \infty\}}{\mathbb{P}_r\{T_{i \wedge \xi} < \infty\}},$$

Passing to the limit we get that

$$\kappa(i, \xi) = \lim_{j \rightarrow \xi} \frac{\mathbb{P}_i\{X_\zeta \in \partial_\infty(j)\}}{\mathbb{P}_r\{X_\zeta \in \partial_\infty(j)\}} = \frac{\mathbb{P}_i\{X_\zeta \in \partial_\infty(\xi(n))\}}{\mathbb{P}_r\{X_\zeta \in \partial_\infty(\xi(n))\}} = \frac{\mathbb{P}_i\{T_{i \wedge \xi} < \infty\}}{\mathbb{P}_r\{T_{i \wedge \xi} < \infty\}}. \quad (3.7)$$

In particular  $\kappa(i, \bullet)$  is the Radon-Nykodim derivative of  $\mathbb{P}_i\{X_\zeta \in \bullet\}$  with respect to  $\mathbb{P}_r\{X_\zeta \in \bullet\}$  (see [11]) so

$$U_{i\xi} = V_{i\xi} + \int_{\partial_\infty} U_{\xi\eta} \kappa(i, \eta) \mathbb{P}_r\{X_\zeta \in d\eta\}.$$

**Remark 3.2** When the tree is recurrent, that is  $V = U$ , the Martin kernel is easily computed as

$$\kappa(i, \eta) = \lim_{j \rightarrow \eta} \frac{V_{ij}}{V_{rj}} = \frac{U_{i\eta}}{w_0}.$$

Therefore,  $\{U_{\bullet\eta}/w_0 : \eta \in \partial_\infty\}$  is the Martin kernel.

## 3.2 Regular and Accessible Points

A close study between  $U$  and the potential  $V$ , in the transient case, needs the description of the regular points on  $\partial_\infty$ . In the classical setting regularity is needed for the continuity up to the boundary for the Dirichlet boundary problem (see for example [13], Theorem 1.23). In our context see Lemma 3.1 (ii) .

**Definition 3.3** A point  $\eta \in \partial_\infty$  is said to be regular if  $\bar{g}(\eta) = 0$ , that is

$$\lim_{j \rightarrow \eta} \mathbb{P}_j\{T_{\partial_r} < \infty\} = 0,$$

and is said to be accessible if it belongs to the closed support of  $\mu$ , that is

$$\mathbb{P}_r\{X_\zeta \in [\eta(n), \infty]\} > 0 \text{ for all } n.$$

If  $\eta$  is not regular we say it is irregular and if it is not accessible we say it is inaccessible. We denote by  $\partial_\infty^{reg}$  the set of regular points and by  $\partial_\infty^{inac}$  the set of inaccessible points.

The classification on accessible and inaccessible points is the same if instead of  $\mathbb{P}_r$ , we use  $\mathbb{P}_i$  for any  $i \in I$ . Similarly  $\eta$  is regular if and only if  $\lim_{j \rightarrow \eta} \mathbb{P}_j\{T_i < \infty\} = 0$  for all  $i \in I$ . From (3.3) this is exactly the case when  $V_{i\eta} = 0$ .

**Lemma 3.1** (i) The measure  $\mu$  concentrates on the set of regular points:  $\mu(\partial_\infty^{reg}) = 1$ .

(ii) A point  $\eta \in \partial_\infty$  is regular if and only if any bounded continuous real function  $f$  defined in  $\partial_\infty \cup \{\partial_r\}$  with  $f(\partial_r) = 0$ , verifies

$$\lim_{j \rightarrow \eta} \mathbb{E}_j(f(X_\zeta)) = f(\eta). \quad (3.8)$$

(iii) Every regular point is accessible.

**Proof.** (i) The function  $\bar{g}(j) = \mathbb{P}_j\{T_{\partial_r} < \infty\}$  is bounded and  $\widehat{Q}$ -harmonic and verifies  $\bar{g}(\partial_r) = 1$ . Using that  $\bar{g}(r) = \mathbb{E}_r(\bar{g}(X_{T_{B^n} \wedge T_{\partial_r}}))$ , the Dominated Convergence Theorem gives

$$\bar{g}(r) = \mathbb{E}_r(\bar{g}(X_\zeta)) = \mathbb{P}_r\{T_{\partial_r} < \infty\} + \int \bar{g}(\xi) \mathbb{P}_r\{X_\zeta \in d\xi\}.$$

From this relation we conclude that  $\bar{g} = 0$   $\mu$ -a.e.. Therefore  $\mu(\partial_\infty^{reg}) = 1$ .

(ii) Since  $f$  is continuous and bounded, for every  $\varepsilon > 0$  fixed there exists  $n$  such that  $|f(\xi) - f(\eta)| \leq \varepsilon$  if  $\xi \in [\eta(n), \infty] \cap \partial_\infty$ . Then for  $j \in [\eta(n), \infty)$  we have

$$|\mathbb{E}_j(f(X_\zeta)) - f(\eta)| \leq 2M\mathbb{P}_j\{T_{\eta(n)} < \zeta\} + 2\varepsilon\mathbb{P}_j\{\zeta \leq T_{\eta(n)}\},$$

where  $M$  is any bound for  $f$ . From this inequality we conclude that

$$\limsup_{j \rightarrow \infty} |\mathbb{E}_j(f(X_\zeta)) - f(\eta)| \leq 2\varepsilon,$$

and then we obtain the desired limit in (3.8).

Conversely, assume now that (3.8) holds for  $f = \mathbf{1}_{\partial_\infty}$  (so  $f(\partial_r) = 0$ ). Then

$$\mathbb{E}_j(f(X_\zeta)) = \mathbb{P}_j\{R_\infty \leq T_{\partial_r}\} = 1 - \mathbb{P}_j\{T_{\partial_r} < \infty\} \xrightarrow{j \rightarrow \eta} 1 = f(\eta),$$

proving that  $\eta$  is regular.

(iii) Let  $\eta$  be a regular point. Take any  $n$  and consider  $f$  the indicator function of  $A = \partial_\infty(\eta(n))$ . For large  $j$  we have  $\mathbb{P}_j\{X_\zeta \in A\} > 0$  which implies  $\mathbb{P}_r\{X_\zeta \in A\} > 0$  and  $\eta$  is accessible.  $\square$

### 3.3 Potential for inaccessible points

We will show that, in the set of inaccessible points, the potential reduces to the recurrent case. For every inaccessible point  $\eta$  we denote by  $N^\eta$  the smallest integer  $n \geq 0$  for which  $\mu(\partial_\infty(\eta(n))) = 0$ . Since  $\{\partial_\infty(\eta(N^\eta)) : \eta \in \partial_\infty^{inac}\}$  is an open cover of  $\partial_\infty^{inac}$ , we can find a finite or countable set  $\{\eta_s : s \in \mathcal{N}\} \subseteq \partial_\infty^{inac}$  such that

$$\partial_\infty^{inac} = \bigsqcup_{s \in \mathcal{N}} (\sqsubset_s \cap \partial_\infty),$$

where  $\sqsubset_s = [m_s, \infty]$  is the infinite tree hanging from  $m_s := \eta_s(N^{\eta_s})$ ,  $s \in \mathcal{N}$ .

**Lemma 3.2** *Let  $j \in \sqsubset_s$  then  $\mathbb{P}_j\{T_{m_s} < \infty\} = \mathbb{P}_j\{T_{m_s} < \zeta\} = 1$ , that is, the restriction of  $U$  to the subtree hanging from  $m_s$  is recurrent. This also implies that every inaccessible point is irregular.*

**Proof.** Observe that  $\mathbb{P}_j$ -a.e. on the set  $\{\zeta \leq T_{m_s}\}$  we have  $X_\zeta \in \sqsubset_s \cap \partial_\infty \subseteq \partial_\infty^{inac}$ . Since  $0 = \mathbb{P}_r\{X_\zeta \in \sqsubset_s\} \geq \mathbb{P}_r\{T_j < \infty\}\mathbb{P}_j\{X_\zeta \in \sqsubset_s\}$ , we conclude  $\mathbb{P}_j\{\zeta \leq T_{m_s}\} = 0$  and the result follows.  $\square$

**Proposition 3.2** *For inaccessible points the potential  $V$  verifies*

$$V|_{\sqsubset_s \times \sqsubset_s} = U|_{\sqsubset_s \times \sqsubset_s} - (U_{m_s m_s} - V_{m_s m_s}) \text{ and } V|_{\sqsubset_s \times \sqsubset_t} \text{ is constant for } s \neq t. \quad (3.9)$$

**Proof.** Consider  $i, j \in \square_s$  we deduce from (3.2) and Lemma 3.2 that

$$U_{ij} - V_{ij} = \int_{\partial_\infty^{reg}} U_{j\eta} \mathbb{P}_i\{X_\zeta \in d\eta\} = \int_{\partial_\infty^{reg}} U_{m_s\eta} \mathbb{P}_{m_s}\{X_\zeta \in d\eta\},$$

which implies the first relation in (3.9). Finally if  $i \in \square_s, j \in \square_t, s \neq t$ , we obtain that

$$V_{ij} = U_{m_s m_t} - \int_{\partial_\infty^{reg}} U_{m_t \eta} \mathbb{P}_{m_s}\{X_\zeta \in d\eta\},$$

which implies the second part in (3.9).  $\square$

**Remark 3.3** Since  $V|_{\square_s \times \square_s}$  is strictly positive and it is equal to  $U|_{\square_s \times \square_s}$  minus a constant, it follows that the potential  $V|_{\square_s \times \square_s}$  is a tree matrix. We recall that this is exactly the case when the tree matrix is recurrent as it is  $U|_{\square_s \times \square_s}$ , see Lemma 3.2.

**Example.** The following example shows that not all accessible points are regular. On figure 1 we have chosen a particular tree, rooted at  $r = 0$ , consisting on a special branch determined by the nodes  $0, 1, 2, \dots$  and subtrees  $T_0, T_1, \dots$ .

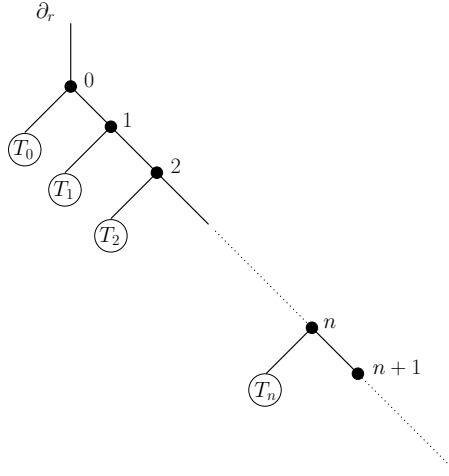


Figure 2.

Each subtree  $T_k$  is regular in the sense that any node  $s \in T_i$  at level  $m$  measured from the root of  $T_k$  (thus at level  $m + k + 1$  measured from  $r$ ) has a constant number of descendants equal to  $s_{m+k+2}$ . The weight function  $w_n$  verifies  $w_0 = 1$  and  $w_{n+1} - w_n = 2^n(w_n - w_{n-1})$ . Also we take  $s_p = 2^p$ . In this way

$$\frac{Q_{ss-}}{(-Q_{ss})} = \frac{(w_{m+k+1} - w_{m+k})^{-1}}{(w_{m+k+1} - w_{m+k})^{-1} + s_{m+k+2}(w_{m+k+2} - w_{m+k+1})^{-1}} = 1/3.$$

The level process on each subtree  $T_i$  is clearly a birth and death chain with birth rate  $2/3$  and death rate  $1/3$ . Therefore  $T_i$  is transient and henceforth

$$\mathbb{P}_0\{X_\zeta \in \partial_\infty(r_i)\} \geq \mathbb{P}_0\{T_{r_i} < \infty\} \mathbb{P}_{r_i}\{X_\zeta \in \partial_\infty(r_i)\} > 0.$$



On the other hand

$$\frac{Q_{k,k-1}}{(-Q_{kk})} = \frac{(w_k - w_{k-1})^{-1}}{(w_k - w_{k-1})^{-1} + 2(w_{k+1} - w_k)^{-1}} = 1 - \frac{1}{2^{n-1} + 1}.$$

This implies that  $\prod_{k=1}^{\infty} \frac{Q_{k,k-1}}{(-Q_{kk})} = a \in (0, 1)$ . Since  $\mathbb{P}_{\eta}\{T_{\partial_r} < \infty\} \geq \frac{1}{w_0} \left( \prod_{k=1}^n \frac{Q_{k,k-1}}{(-Q_{kk})} \right) \geq \frac{a}{w_0}$ , the point  $\eta \in \partial_{\infty}$  determined by the special branch, is irregular but accessible.

### 3.4 The Kernel at the Boundary is a Filtered Operator

Let us introduce the operator  $W$ , acting on  $L^p(\mu)$ , with kernel  $U$ . We point out that  $U$  and  $W$  acting on  $\partial_{\infty}$  where introduced in [28] section 4, and they are used in [29] section 2.3 to study the capacity function on the boundary.

**Definition 3.4** *For any (positive) bounded, real and measurable function  $f$  with domain in  $\partial_{\infty}$  we define*

$$Wf(\eta) = \int_{\partial_{\infty}} U_{\eta\xi} f(\xi) \mu(d\xi)$$

*which is also a (positive) real and measurable function.*

We notice that the integral defining  $W$  can be made over  $\partial_{\infty}$  or  $\partial_{\infty}^{reg}$ , because this last set is of full measure  $\mu$ . We have from (3.4) and  $w_0 = U_{r\eta}$  that

$$W\mathbf{1}(\eta) = \int_{\partial_{\infty}} U_{\eta\xi} \mu(d\xi) = \frac{w_0 - V_{r\eta}}{\mathbb{P}_r\{X_{\zeta} \in \partial_{\infty}\}}.$$

Then  $Wf$  is bounded for any bounded  $f$ . Since  $V_{r\eta} = 0$  for any regular point  $\eta$ , we conclude that  $W\mathbf{1}$  is constant  $\mu$ -a.e., where this constant, denoted by  $\alpha$ , is given by  $\alpha = w_0/\mathbb{P}_r\{X_{\zeta} \in \partial_{\infty}\}$ . In general we have  $W\mathbf{1} \leq \alpha$  in  $\partial_{\infty}$ .

The action of  $W$  on measures is given by  $\nu W(A) = \int W\mathbf{1}_A(\xi) \nu(d\xi)$ . It is direct to see that  $\mu W = \alpha\mu$ . Then  $\alpha^{-1}W$  is a Markov operator preserving  $\mu$ . Hence, for every  $p \geq 1$ , the operator  $W : L^p(\mu) \rightarrow L^p(\mu)$  is well defined,  $\|W\|_p = \alpha$  and  $W$  is self adjoint in  $L^2(\mu)$ .

Recall notations  $\partial_{\infty}(i) = [i, \infty] \cap \partial_{\infty}$  made in (1.4) and  $geod(r, \xi) = (\xi(k) : k \in \mathbb{N})$  for  $\xi \in \partial_{\infty}$ . We put

$$C^k(\xi) = \partial_{\infty}(\xi(k)) = \{\eta \in \partial_{\infty} : \xi(k) = \eta(k)\}.$$

We also consider

$$\Delta_k(w) = w_k - w_{k-1} \text{ for } k \in \mathbb{N}, \Delta_{-1}(w) = 0.$$

Notice that  $\Delta_k(w) > 0$  for  $k \in \mathbb{N}$ . For  $f \in L^1(\mu)$  it is verified

$$Wf(\eta) = \sum_{k \in \mathbb{N}} w_k \int_{C^k(\eta) \setminus C^{k+1}(\eta)} f d\mu = \sum_{k \in \mathbb{N}} \Delta_k(w) \int_{C^k(\eta)} f d\mu. \quad (3.10)$$

The set function  $C^k$ , with domain  $\partial_\infty$ , takes a finite number of values. We denote by  $\mathcal{F}_k$  the  $\sigma$ -field in  $\partial_\infty$  generated by the sets  $(C^k)$ . This sequence of  $\sigma$ -fields is increasing and generating, that is  $\mathcal{F}_\infty = \sigma(\mathbf{T})$ . Thus,  $\mathcal{F} = (\mathcal{F}_k : k \in \mathbb{N})$  is a generating filtration in  $\partial_\infty$ . With this notation equality (3.10) can be written as

$$Wf(\bullet) = \sum_{k \in \mathbb{N}} \Delta_k(w) \mu(C^k(\bullet)) \mathbb{E}_\mu(f | \mathcal{F}_k)(\bullet). \quad (3.11)$$

Now, consider on  $\partial_\infty$  the following process

$$G = (G_n : n \in \mathbb{N}) \text{ where } G_n(\eta) = \sum_{k \geq n} \Delta_k(w) \mu(C^k(\eta)). \quad (3.12)$$

Since  $G_0 = W\mathbf{1} \leq \alpha$  we obtain that  $G_0$  is a convergent series. On the other hand, since every regular point is accessible we conclude that  $\mu(C^k(\xi)) > 0$  for every  $k \in \mathbb{N}$ ,  $\xi \in \partial_\infty^{reg}$  and in particular  $G_n > 0$ ,  $\mu$ -a.e. for every  $n \in \mathbb{N}$ . We also have

$$G_n(\eta) = G_0 - \sum_{k=0}^{n-1} \Delta_k(w) \mu(C^k(\eta)) \text{ is } \mathcal{F}_{n-1} \text{ measurable.}$$

Therefore if  $|\xi \wedge \eta| \geq n$  we have  $G_i(\eta) - G_{i+1}(\eta) = G_i(\xi) - G_{i+1}(\xi)$ ,  $i = 0, \dots, n$ . Moreover, if  $\xi, \eta$  are regular points then  $G_0(\eta) = G_0(\xi) = \alpha$  and

$$G_i(\xi) = G_i(\eta), \text{ for all } i \leq |\xi \wedge \eta|. \quad (3.13)$$

The process  $(G_n)$  is  $\mathcal{F}$ -predictable, positive, bounded by  $\alpha$  and decreasing to 0 as  $n \rightarrow \infty$ . Then  $G_n \mathbb{E}_\mu(\cdot | \mathcal{F}_n)$  converges to 0 in  $L^p(\mu)$  for every  $p \in [1, \infty]$ . Therefore, integration by parts on (3.11) gives

$$W = \sum_{n \in \mathbb{N}} (G_n - G_{n+1}) \mathbb{E}_\mu(\cdot | \mathcal{F}_n) = \sum_{n \in \mathbb{N}} G_n (\mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1})). \quad (3.14)$$

This equality being in the sense of operators. Thus, we have shown the following result.

**Proposition 3.3** *The self adjoint operator  $W$  acting on  $L^2(\mu)$  is an stochastic integral operator (or a filtered operator), that is, there exists a filtration  $\mathcal{F} = (\mathcal{F}_n)$  and  $G = (G_n)$  a  $\mathcal{F}$ -predictable process, such that  $W = \sum_{n \in \mathbb{N}} G_n (\mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1}))$ .*

For definitions and properties of stochastic integral operators see [17], and for its characterization in the countable case see [14].

Let us consider  $\mathcal{D} = \cup_{n \in \mathbb{N}} L^2(\mathcal{F}_n, \mu)$  the set of simple functions over the algebra  $\cup_{n \in \mathbb{N}} \mathcal{F}_n$ . Clearly  $\mathcal{D}$  is a dense subset in  $L^2(\mu)$ . Notice that the operator  $L = \sum_{n \in \mathbb{N}} G_n^{-1} (\mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1}))$  is well defined in  $\mathcal{D}$ . As  $G_n$  is  $\mathcal{F}_{n-1}$  measurable the following equalities hold on  $\mathcal{D}$ ,

$$LW = WL = \sum_{n \in \mathbb{N}} \mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1}) = \mathbb{I}_{\mathcal{D}}.$$

Here  $\mathbb{I}_{\mathcal{D}}$  is the identity on  $\mathcal{D}$ . In particular,  $Im(W) = W(L^2(\mu))$  contains  $\mathcal{D}$ , so  $Im(W)$  is dense in  $L^2(\mu)$ . Since  $W$  is a self adjoint operator, we get that  $W$  is one-to-one. Hence we can extend  $L$  to  $Im(W)$  by  $Lg = f$  for  $g \in Im(W)$ ,  $g = Wf$ . Therefore

$$WL = \mathbb{I}_{Im(W)}, \quad LW = \mathbb{I}_{L^2(\mu)}.$$

We put  $L = W^{-1}$  and we assume implicitly that its domain is  $Im(W)$ , so

$$W^{-1} = \sum_{n \in \mathbb{N}} G_n^{-1} (\mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1})). \quad (3.15)$$

Observe that  $W^{-1}\mathbf{1} = \alpha^{-1} \mu$ -a.e.. The operator  $-W^{-1}$  is a generator of a subMarkov kernel defined in the boundary, that will be studied in section 6.

Let us compute  $W^{-1}$  in  $\mathcal{D}$ . Fix a set  $C^n \in \mathcal{F}_n$ . For  $k \leq n$  we denote by  $C^k$  the element in  $\mathcal{F}_k$  such that  $C^n \subseteq C^k$ . We also put  $C^{-1} = \phi$ . From (3.15) we obtain

$$\begin{aligned} W^{-1}\mathbf{1}_{C^n} &= \sum_{k=0}^n G_k^{-1} (\mathbb{E}_\mu(\mathbf{1}_{C^n} | \mathcal{F}_k) - \mathbb{E}_\mu(\mathbf{1}_{C^n} | \mathcal{F}_{k-1})) \\ &= \sum_{k=0}^n G_k^{-1} \left( \frac{\mu(C^n)}{\mu(C^k)} \mathbf{1}_{C^k} - \frac{\mu(C^n)}{\mu(C^{k-1})} \mathbf{1}_{C^{k-1}} \right) = G_n^{-1} \mathbf{1}_{C^n} + \sum_{k=0}^{n-1} (G_k^{-1} - G_{k+1}^{-1}) \frac{\mu(C^n)}{\mu(C^k)} \mathbf{1}_{C^k}. \end{aligned} \quad (3.16)$$

Let  $\eta, \xi \in \partial_\infty$ ,  $\eta \neq \xi$ , and take  $n > |\eta \wedge \xi|$ . Since  $C^k(\xi) = C^k(\eta)$  for  $k \leq |\eta \wedge \xi|$  we get

$$W^{-1}\mathbf{1}_{C^n(\eta)}(\xi) = \sum_{k=0}^{|\eta \wedge \xi|} (G_k^{-1}(\eta) - G_{k+1}^{-1}(\eta)) \frac{\mu(C^n(\eta))}{\mu(C^k(\eta))}. \quad (3.17)$$

Then

$$\frac{W^{-1}\mathbf{1}_{C^n(\eta)}(\xi)}{\mu(C^n(\eta))} = - \sum_{k=0}^{|\eta \wedge \xi|} \frac{\Delta_k(w)}{G_k(\eta)G_{k+1}(\eta)}.$$

Thus, for  $\xi \neq \eta$  the following limit exists

$$W^{-1}(\xi, \eta) = \lim_{n \rightarrow \infty} \frac{W^{-1}\mathbf{1}_{C^n(\eta)}(\xi)}{\mu(C^n(\eta))} = - \sum_{k=0}^{|\eta \wedge \xi|} \frac{\Delta_k(w)}{G_k(\eta)G_{k+1}(\eta)} < 0. \quad (3.18)$$

**Remark 3.4** *The operator*

$$\underline{\mathbf{W}}^{-1} := W^{-1} - G_0^{-1} \mathbb{E}_\mu = \sum_{n \geq 1} G_n^{-1} (\mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1})) \quad (3.19)$$

verifies  $\underline{\mathbf{W}}^{-1} \mathbf{1} = 0$ , and  $-\underline{\mathbf{W}}^{-1}$  is a generator of a Markov process in the boundary.

In the next result we explicit the Dirichlet form associated to  $-W^{-1}$ . More precisely, we get the Beurling-Deny formula following closely the construction done in [20] (see Theorem 3.2.1). We compute it for simple functions using mainly the fact that  $(G_n)$  is predictable. Then it can be extended by density arguments.

**Proposition 3.4** *Let  $\mathbf{E}(f, g) = \int_{\partial_\infty} g W^{-1} f d\mu$  be the Dirichlet symmetric form associated to  $-W^{-1}$  in  $L^2(\mu)$ . Let  $D = \{(\eta, \eta) : \eta \in \partial_\infty\}$  be the diagonal in  $\partial_\infty^2$ . Then for all  $f, g \in \mathcal{D}$ , the set of simple functions, we have*

$$\mathbf{E}(f, g) = \frac{1}{2} \int_{\partial_\infty \times \partial_\infty \setminus D} (f(\eta) - f(\xi))(g(\eta) - g(\xi)) H(\eta, \xi) \mu \otimes \mu(d\eta, d\xi) + \frac{1}{G_0} \int_{\partial_\infty} f(\eta) g(\eta) \mu(d\eta),$$

where

$$H = \sum_{n \geq 0} \sum_{j \in B^n} \frac{1}{\mu(C_j)} \left( \frac{1}{G_{n+1}} - \frac{1}{G_n} \right) \mathbf{1}_{C_j \times C_j}, \quad (3.20)$$

with  $C_j = \partial_\infty(j)$ .

**Proof.** We notice that  $H(\xi, \eta)$  in (3.20) is well defined for  $\xi \neq \eta$  and it is symmetric because  $G_{n+1}, G_n$  are constant over  $C_j$  for  $j \in B^n$ .

We denote by  $\mathbb{E}_n = \mathbb{E}_\mu(\cdot | \mathcal{F}_n)$  and by  $\langle \cdot, \cdot \rangle$  the inner product in  $L^2(\mu)$ . We follow the construction of  $\mathbf{E}$  given in [20].

The resolvent  $R_\beta = \int_0^\infty e^{-\beta t} e^{-tW^{-1}} dt$  is given by

$$\beta R_\beta = \sum_{n \geq 0} \frac{\beta G_n}{\beta G_n + 1} (\mathbb{E}_n - \mathbb{E}_{n-1}) = \sum_{n \geq 0} h_n^{(\beta)} \mathbb{E}_n,$$

where  $h_n^{(\beta)} = \frac{\beta G_n}{\beta G_n + 1} - \frac{\beta G_{n+1}}{\beta G_{n+1} + 1} \in \mathcal{F}_n$ .

We have

$$\begin{aligned} \langle f, \beta R_\beta g \rangle &= \sum_{n \geq 0} \int f h_n^{(\beta)} \mathbb{E}_n g d\mu = \sum_{n \geq 0} \int h_n^{(\beta)} \mathbb{E}_n(f) \mathbb{E}_n(g) d\mu \\ &= \sum_{n \geq 0} \sum_{j \in B^n} \int_{C_j} h_n^{(\beta)} \left[ \int_{C_j} f(\xi) \mu(d\xi) / \mu(C_j) \int_{C_j} g(\eta) \mu(d\eta) / \mu(C_j) \right] d\mu \\ &= \sum_{n \geq 0} \int f(\xi) g(\eta) H_n^{(\beta)}(\xi, \eta) \mu \otimes \mu(d\xi, d\eta), \end{aligned}$$

where  $H_n^{(\beta)} = \sum_{n \geq 0} \sum_{j \in B^n} \frac{1}{\mu(C_j)} \left( \frac{\beta G_n}{\beta G_n + 1} - \frac{\beta G_{n+1}}{\beta G_{n+1} + 1} \right) \mathbf{1}_{C_j \times C_j}$ . Outside the diagonal we have that

$$\beta H_n^{(\beta)}(\xi, \eta) \xrightarrow{\beta \rightarrow \infty} H_n(\xi, \eta).$$

Since for  $f, g \in \mathcal{D}$  with disjoint support we have

$$\mathbf{E}(f, g) = \lim_{\beta \rightarrow \infty} -\beta \langle f, \beta R_\beta g \rangle = - \int f(\xi) g(\eta) H(\xi, \eta) \mu \otimes \mu(d\xi, d\eta).$$

Then, the result holds in this case.

The only thing left to compute is  $\mathbf{E}(\mathbf{1}_C, \mathbf{1}_C)$ , for any  $C$  an atom of some  $\mathcal{F}_n, n \geq 0$ . This is done by linearity and the following fact, which is direct to show

$$\mathbf{E}(\mathbf{1}_C, \mathbf{1}) = \frac{1}{G_0} \int \mathbf{1}_C^2 d\mu.$$

□

Hence, the diffusive part in the Beurling-Deny formula vanishes, so the subMarkov process associated to  $-W^{-1}$ , is a pure jump process. This conclusion can be also obtained directly by using the arguments developed in [2] Theorem 4.1.

**Remark 3.5** Let  $\underline{\mathbf{E}}(f, g) = \int_{\partial_\infty} g \underline{\mathbf{W}}^{-1} f d\mu$  be the Dirichlet symmetric form associated to  $-\underline{\mathbf{W}}^{-1}$  in  $L^2(\mu)$ . Then for all  $f, g \in \mathcal{D}$ , the set of simple functions, we have

$$\underline{\mathbf{E}}(f, g) = \frac{1}{2} \int_{\partial_\infty \times \partial_\infty \setminus D} (f(\eta) - f(\xi))(g(\eta) - g(\xi)) H(\eta, \xi) \mu \otimes \mu(d\eta, d\xi),$$

that is, in this case the killing part disappears, as it must happen by construction of  $-\underline{\mathbf{W}}^{-1}$ .

### 3.5 The Martin Kernel for Accessible Points

From (3.2) the Martin kernel for an irregular point  $\xi$  is given by

$$\kappa(i, \xi) = \frac{U_{i\xi} - \int_{\partial_\infty} U_{\eta\xi} \mathbb{P}_i\{X_\zeta \in d\eta\}}{w_0 - \int_{\partial_\infty} U_{\eta\xi} \mathbb{P}_r\{X_\zeta \in d\eta\}}. \quad (3.21)$$

The study of the Martin kernel for regular points needs an extra work because numerator and denominator vanish. This constitutes the main object of this section. Next formulae relate the operator  $W$  and the exit measure.

**Proposition 3.5** For any  $i, j \in I$  we have

$$\mathbb{P}_i\{X_\zeta \in \partial_\infty(j)\} = \int_{\partial_\infty} U_{i\xi}(W^{-1}\mathbf{1}_{\partial_\infty(j)})(\xi) \mu(d\xi). \quad (3.22)$$

**Proof.** The function  $h_1(i) = \mathbb{P}_i\{X_\zeta \in \partial_\infty(j)\}$  is harmonic and bounded. Moreover for any regular point  $\eta$  we have

$$\lim_{i \rightarrow \eta} \mathbb{P}_i\{X_\zeta \in \partial_\infty(j)\} = \mathbf{1}_{\partial_\infty(j)}(\eta),$$

which implies that  $\lim h_1 = \mathbf{1}_{\partial_\infty(j)} \mu$ -a.e..

On the other hand, consider  $h_2(i) := \int_{\partial_\infty} U_{i\xi}(W^{-1}\mathbf{1}_{\partial_\infty(j)})(\xi)\mu(d\xi)$ . This function is also harmonic because for every  $\xi \in \partial_\infty$  the function  $U_{i\xi}$  is harmonic on  $I$ . Let us show that  $h_2$  is a bounded function. From (3.17) one checks that  $\|W^{-1}\mathbf{1}_{\partial_\infty(j)}\|_\infty < \infty$ . Then

$$|h_2(i)| \leq \|W^{-1}\mathbf{1}_{\partial_\infty(j)}\|_\infty \int_{\partial_\infty} U_{\eta\xi}\mu(d\xi) = \|W^{-1}\mathbf{1}_{\partial_\infty(j)}\|_\infty W\mathbf{1}(\eta) < \infty,$$

where  $\eta$  is any point in  $\partial_\infty(i)$ . Hence  $h_2$  is bounded. Finally, by the Dominated Convergence Theorem we conclude the pointwise convergence

$$\lim_{i \rightarrow \eta} h_2(i) = \int_{\partial_\infty} U_{\eta\xi}(W^{-1}\mathbf{1}_{\partial_\infty(j)})(\xi)\mu(d\xi).$$

The result follows from the equality  $\int_{\partial_\infty} U_{\eta\xi}(W^{-1}\mathbf{1}_{\partial_\infty(j)})(\xi)\mu(d\xi) = \mathbf{1}_{\partial_\infty(j)}(\eta) \mu$ -a.e. in  $\eta \in \partial_\infty$ .  $\square$

**Corollary 3.1** *Let  $h : I \rightarrow \mathbb{R}$  be a harmonic function such that  $\lim h = \varphi \mu$ -a.e. (for example if  $h$  is bounded). Assume  $\varphi$  is a simple function, that is  $\varphi \in \mathcal{D}$  (in particular  $\varphi$  is in the domain of  $W^{-1}$ ). Then for all  $i \in I$*

$$h(i) = \int U_{i\xi}(W^{-1}\varphi)(\xi)\mu(d\xi). \quad (3.23)$$

**Proof.** It is direct from (3.22) by decomposing  $\varphi$  as a finite linear combination of indicator functions based on the sets  $C^{n_1}(\eta_1), \dots, C^{n_k}(\eta_k)$ .  $\square$

**Remark 3.6** *Then, in a "dense" class of harmonic functions we have the representation  $h(i) = \int U_{i\xi}d\nu(\xi)$  with  $d\nu(\xi) = W^{-1}\varphi(\xi)\mu(d\xi)$ . This representation is similar to the one using the Martin kernel as in [11]. Nevertheless, there are some differences. Even if  $h$  is positive,  $\nu$  may be a signed measure. On the other hand the characterization  $d\nu = W^{-1}\varphi d\mu$  gives additional information on this signed measure. We recall that in the Martin representation,  $\varphi$  is the Radon-Nikodym derivative of the absolute continuous part of the representing measure with respect to  $\mu$  (see for example [39]).*

Recall that a real function  $f$  is increasing in the tree, which we denote by  $\preceq$  -increasing, if  $i \preceq j$  implies  $f(i) \leq f(j)$ .

**Theorem 3.1** *A function  $h : I \rightarrow \mathbb{R}_+$  is harmonic and  $\preceq$ -increasing if and only there exists a finite (nonnegative) measure  $\nu$  in  $\partial_\infty$  such that*

$$h(i) = \int_{\partial_\infty} U_{i\xi} d\nu(\xi) \text{ for every } i \in I. \quad (3.24)$$

**Proof.** If  $h$  verifies (3.24) then it is harmonic and increasing since  $U_{\bullet\xi}$  is so. Let us assume now that  $h$  is a nonnegative harmonic and increasing function. From Proposition 2.4 proven for finite matrices we get

$$\forall n \exists! \alpha^{(n)} : B^n \rightarrow \mathbb{R} \text{ such that if } |i| \leq n : h(i) = \sum_{j \in B^n} U_{ij} \alpha^{(n)}(j).$$

In particular if  $|i| = n - 1$  we find

$$h(i^+) = \sum_{j \in B^n} U_{i+j} \alpha^{(n)}(j) = \sum_{j \in B^n, j \neq i^+} U_{ij} \alpha^{(n)}(j) + U_{i+i^+} \alpha^{(n)}(i^+) = h(i) + (U_{i+i^+} - U_{ii^+}) \alpha^{(n)}(i^+).$$

Therefore

$$\alpha^{(n)}(i^+) = \frac{h(i^+) - h(i)}{U_{i+i^+} - U_{ii^+}},$$

and  $\alpha^{(n)}$  is a measure in  $B^n$ . Let us show that these measures verify the consistence property. We have

$$\text{for } |i| \leq n : h(i) = \sum_{j \in \tilde{B}^{n+1}} U_{ij} \alpha^{(n+1)}(j) = \sum_{k \in B^n} U_{ik} \left( \sum_{j \in S_k} \alpha^{(n+1)}(j) \right) = \sum_{k \in B^n} U_{ik} \alpha^{(n)}(k).$$

From uniqueness of  $\alpha^{(n)}$  we deduce  $\alpha^{(n)}(k) = \sum_{j \in S_k} \alpha^{(n+1)}(j)$ . Then the consistence property is verified. The total mass of  $\alpha^{(n)}$  is given by  $h(r) = w_0 \sum_{j \in B^n} \alpha^{(n)}(j)$ . Then there exists a finite measure in the boundary such that  $h(i) = \int_{\partial_\infty} U_{i\xi} d\nu(\xi)$ , for  $i \in I$ .  $\square$

**Remark 3.7** *The measure  $\nu$  in the previous result can be singular with respect to  $\mu$ . For example, it is enough to take  $\xi$  an inaccessible point and consider the function  $h(i) = U_{i\xi}$ , which is represented by the measure  $\nu = \delta_\xi$ .*

The next result is a representation as an integral of  $U$ , of all harmonic functions that satisfies a certain finite variation condition.

**Theorem 3.2** *Assume that  $h : I \rightarrow \mathbb{R}$  is a bounded harmonic function. Then, there exists a finite signed measure  $\nu$  such that*

$$h(i) = \int_{\partial_\infty} U_{i\xi} d\nu(\xi) \text{ for every } i \in I, \quad (3.25)$$

if and only if the following condition holds

$$\sup_{n \geq 1} \frac{1}{w_n - w_{n-1}} \sum_{j \in B^n} |h(j) - h(j^-)| < \infty. \quad (3.26)$$

In particular if this condition holds then  $h = h^+ - h^-$  is the difference of two increasing nonnegative harmonic functions  $h^+, h^-$  given by the positive and negative part of  $\nu$ .

**Proof.** Let us first assume that  $h$  is strictly positive. If (3.25) holds, then

$$h(i) - h(i^-) = \int (U_{i\xi} - U_{i-\xi}) d\nu(\xi) = (U_{ii} - U_{i-i}) \nu(\partial_\infty(i)),$$

from which we obtain

$$|h(i) - h(i^-)| \leq (w_n - w_{n-1}) |\nu|(\partial_\infty(i)).$$

Summing over  $B^n$  this inequality yields

$$\frac{1}{w_n - w_{n-1}} \sum_{i \in B^n} |h(i) - h(i^-)| \leq |\nu|(\partial_\infty) < \infty.$$

Let us now assume that (3.26) holds. As in the proof of Theorem 3.1 we have that for all  $n$  and all  $i \in I$  such that  $|i| \leq n$

$$h(i) = \sum_{j \in B^n} U_{ij} \alpha^{(n)}(j),$$

where

$$\alpha^{(n)}(j) = \frac{h(j) - h(j^-)}{U_{jj} - U_{jj^-}} = \frac{h(j) - h(j^-)}{w_n - w_{n-1}}.$$

Let us define the signed measure  $\nu_n$  by  $\nu_n(\partial_\infty(j)) = \alpha^{(n)}(j)$ . Then we obtain that  $\nu_n(\partial_\infty) = h(r)/w_0 > 0$  and

$$\sup_n |\nu_n|(\partial_\infty) < \infty.$$

Therefore, there exists a subsequence  $(\nu_{n_k})$  converging weakly to a finite signed measure  $\nu \neq 0$ . Moreover,  $\nu(\partial_\infty) = h(r)/w_0$  and since  $U_{i\bullet}$  is a bounded continuous function we get

$$h(i) = \lim_k \int U_{i\xi} d\nu_{n_k}(\xi) = \int U_{i\xi} d\nu(\xi)$$

For the general case remind that  $\ell(i) =: 1 - \bar{g}(i) = \mathbb{P}_i(X_\zeta \in \partial_\infty)$  is a nonnegative harmonic function.  $\ell$  is also an increasing harmonic function with limit 1 in the boundary, then

$$\ell(i) = \int U_{i\xi}(W^{-1}\mathbf{1})(\xi) d\mu(\xi) = \frac{\ell(r)}{w_0} \int U_{i\xi} d\mu(\xi) = \int U_{i\xi} d\nu(\xi),$$



where  $\nu$  is the finite measure  $\frac{\ell(r)}{w_0}\mu$ . Since  $\ell(i) \geq \ell(r) > 0$  we can take a large constant  $C$  such that the function  $\bar{h} = h + C\ell$  is a nonnegative bounded harmonic function. It is direct to check that  $h$  satisfies (3.26) if and only if  $\bar{h}$  satisfies it, from where the result holds.  $\square$

We notice that, since  $h$  is harmonic we have

$$\frac{1}{w_n - w_{n-1}}(h(j) - h(j^-)) = Q_{jj^-}(h(j) - h(j^-)) = \sum_{j^+} Q_{jj^+}(h(j^+) - h(j)).$$

Then,

$$\frac{1}{w_n - w_{n-1}}|h(jn) - h(j^-)| \leq \sum_{j^+} \frac{1}{w_{n+1} - w_n}|h(j^+) - h(j)|,$$

implying that  $\frac{1}{w_n - w_{n-1}} \sum_{j \in B^n} |h(j) - h(j^-)|$  is monotone in  $n$ .

Let us give a formula for the Martin kernel in terms of  $U$  and  $\mu$ . For this reason we first prove the following result.

**Proposition 3.6** *For  $\eta \in \partial_\infty$ ,  $i \in I$ ,  $n \geq 1$  we have*

$$\begin{aligned} \mathbb{P}_i\{X_\zeta \in C^n(\eta)\} &= \mu(C^n(\eta)) \left[ \frac{U_{i\eta}}{G_{|i \wedge \eta|+1}(\eta)} \mathbf{1}_{I \setminus [\eta(n), \infty)}(i) + \frac{1}{G_n(\eta)} \mathbb{E}(U_{i\bullet} | \mathcal{F}_n)(\eta) \mathbf{1}_{[\eta(n), \infty)}(i) \right. \\ &\quad \left. + \sum_{k=0}^{n-1} \left( \frac{1}{G_k(\eta)} - \frac{1}{G_{k+1}(\eta)} \right) \mathbb{E}(U_{i\bullet} | \mathcal{F}_k)(\eta) \mathbf{1}_{[\eta(k), \infty)}(i) \right]. \end{aligned}$$

*In particular, if  $\eta$  is accessible and  $n > |i \wedge \eta|$  we get*

$$\frac{\mathbb{P}_i\{X_\zeta \in C^n(\eta)\}}{\mu(C^n(\eta))} = \frac{U_{i\eta}}{G_{|i \wedge \eta|+1}(\eta)} + \sum_{k=0}^{|i \wedge \eta|} \left( \frac{1}{G_k(\eta)} - \frac{1}{G_{k+1}(\eta)} \right) \mathbb{E}(U_{i\bullet} | \mathcal{F}_k)(\eta). \quad (3.27)$$

**Proof.** Let  $\eta$  and  $n$  be fixed. We denote  $C^k = C^k(\eta) = \partial_\infty(\eta(k))$  and  $A^k = [\eta(k), \infty)$ . From (3.22) we have

$$h_\eta(i) := \mathbb{P}_i\{X_\zeta \in C^n\} = \int_{\partial_\infty} U_{i\xi}(W^{-1}\mathbf{1}_{C^n})(\xi) \mu(d\xi).$$

Now, let us compute

$$\rho^k(i) := \int_{\partial_\infty} U_{i\xi} \mathbf{1}_{C^k}(\xi) \mu(d\xi).$$

We examine two different cases. If  $i \notin A^k$  then  $U_{i\xi} = U_{i\eta(k)}$  for every  $\xi \in C^k$ , and so  $\rho^k(i) = U_{i\eta(k)} \mu(C^k)$ . If  $i \in A^k$  then  $\rho^k(i) = \sum_{\substack{j \in A^k \\ |j|=|i|}} U_{ij} \mu(\partial_\infty(j))$ . We summarize these relations in

$$\rho^k(i) = U_{i\eta(k)} \mu(C^k) \mathbf{1}_{I \setminus A^k}(i) + \sum_{\substack{j \in A^k \\ |j|=|i|}} U_{ij} \mu(\partial_\infty(j)) \mathbf{1}_{A^k}(i). \quad (3.28)$$

Now we use (3.16) to get

$$\begin{aligned} \int U_{i\xi}(W^{-1}\mathbf{1}_{C^n})(\xi)\mu(d\xi) &= \frac{1}{G_n} \left[ U_{i\eta(n)}\mu(C^n) \mathbf{1}_{I\setminus A^n}(i) + \sum_{\substack{j \in A^n \\ |j|=|i|}} U_{ij} \mu(\partial_\infty(j)) \mathbf{1}_{A^n}(i) \right] \\ &+ \sum_{k=0}^{n-1} \left( \frac{1}{G_k} - \frac{1}{G_{k+1}} \right) \frac{\mu(C^n)}{\mu(C^k)} \left[ U_{i\eta(k)}\mu(C^k) \mathbf{1}_{I\setminus A^k}(i) + \sum_{\substack{j \in A^k \\ |j|=|i|}} U_{ij} \mu(\partial_\infty(j)) \mathbf{1}_{A^k}(i) \right]. \end{aligned}$$

From

$$\mathbb{E}(U_{i\bullet}|\mathcal{F}_n)(\eta) = \begin{cases} U_{i\eta(k)} & \text{if } i \notin A^k \\ \sum_{\substack{j \in A^n \\ |j|=|i|}} U_{ij}\mu(\partial_\infty(j)) & \text{if } i \in A^k, \end{cases}$$

to get

$$\begin{aligned} &\int U_{i\xi}(W^{-1}\mathbf{1}_{C^n})(\xi)\mu(d\xi) \\ &= \mu(C^n) \left[ \sum_{k=0}^{n-1} \left( \frac{1}{G_k} - \frac{1}{G_{k+1}} \right) \mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta) \mathbf{1}_{A^k}(i) + \frac{1}{G_n} \mathbb{E}(U_{i\bullet}|\mathcal{F}_n)(\eta) \mathbf{1}_{A^n}(i) \right] \\ &+ \mu(C^n) \left[ \sum_{k=0}^{n-1} \left( \frac{1}{G_k} - \frac{1}{G_{k+1}} \right) U_{i\eta(k)} \mathbf{1}_{I\setminus A^k}(i) + \frac{1}{G_n} U_{i\eta(n)} \mathbf{1}_{I\setminus A^n}(i) \right]. \end{aligned}$$

Now  $i \in I \setminus A^k$  implies  $k > |i \wedge \eta|$ . Since for  $k \geq |i \wedge \eta|$  we have  $U_{i\eta(k)} = U_{i\eta}$ , we can simplify the last term in the previous equation to

$$\frac{\mu(C^n)}{G_{|i \wedge \eta|+1}} U_{i\eta} \mathbf{1}_{I\setminus A^n}(i).$$

Then we get

$$\begin{aligned} \mathbb{P}_i\{X_\zeta \in C^n(\eta)\} &= \mu(C^n) \left[ \frac{U_{i\eta}}{G_{|i \wedge \eta|+1}} \mathbf{1}_{I\setminus A^n}(i) + \frac{1}{G_n} \mathbb{E}(U_{i\bullet}|\mathcal{F}_n)(\eta) \mathbf{1}_{A^n}(i) + \right. \\ &\quad \left. \sum_{k=0}^{n-1} \left( \frac{1}{G_k(\eta)} - \frac{1}{G_{k+1}(\eta)} \right) \mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta) \mathbf{1}_{A^k}(i) \right]. \end{aligned}$$

□

**Theorem 3.3** *Let  $i \in I$  and  $\eta$  be an accessible point. Then the Martin kernel has the representation*

$$\kappa(i, \eta) = \frac{1}{\mathbb{P}_r\{X_\zeta \in \partial_\infty\}} \sum_{k=0}^{|i \wedge \eta|+1} \frac{1}{G_k(\eta)} (\mathbb{E}(U_{i\bullet}|\mathcal{F}_k)(\eta) - \mathbb{E}(U_{i\bullet}|\mathcal{F}_{k-1})(\eta)), \quad (3.29)$$

where by convention  $\mathbb{E}(\cdot|\mathcal{F}_{-1}) = 0$ .

**Proof.** We use Proposition 3.6 and the equality  $\frac{U_{r\eta}}{G_0} = \frac{w_0}{G_0} = \mathbb{P}_r\{X_\zeta \in \partial_\infty\}$  to get

$$\kappa(i, \eta) = \frac{1}{\mathbb{P}_r\{X_\zeta \in \partial_\infty\}} \left[ \frac{U_{i\eta}}{G_{|i \wedge \eta|+1}(\eta)} + \sum_{k=0}^{|i \wedge \eta|} \left( \frac{1}{G_k(\eta)} - \frac{1}{G_{k+1}(\eta)} \right) \mathbb{E}(U_{i\bullet} | \mathcal{F}_k)(\eta) \right],$$

and the result follows.  $\square$

**Corollary 3.2** *For  $i \in I$  fixed, the Martin kernel  $\kappa(i, \bullet)$  is the image of  $U_{i\bullet}$  by an stochastic integral operator, in fact*

$$\kappa(i, \eta) = \sum_{k=0}^{\infty} \tilde{G}_k^{(i)}(\eta) (\mathbb{E}(U_{i\bullet} | \mathcal{F}_k) - \mathbb{E}(U_{i\bullet} | \mathcal{F}_{k-1}))(\eta),$$

where  $\tilde{G}^{(i)} = (\tilde{G}_k^{(i)} : k \in \mathbb{N})$  is a  $\mathcal{F}$ -predictable process.

**Proof.** It suffices to take  $\tilde{G}_k^{(i)} = \mathbf{1}_{D_k^{(i)}} \mathbb{P}_r\{X_\zeta \in \partial_\infty\}^{-1} G_k^{-1}$ , where  $D_k^{(i)} = \{\xi \in \partial_\infty : \xi \wedge i \geq k-1\}$  is a  $\mathcal{F}_{k-1}$ -measurable set.  $\square$

Let us revisit the Martin kernel for an irregular point  $\eta$ . From (3.7), if  $\xi \in \partial_\infty^{reg}$  the kernel  $\kappa(i, \xi)$  is the Radon-Nykodim derivative of  $\mathbb{P}_i\{X_\zeta \in d\xi\}$  with respect to  $\mathbb{P}_r\{X_\zeta \in d\xi\}$ . Therefore if  $\eta$  is an accessible irregular point we obtain from (3.21)

$$\kappa(i, \eta) = \frac{U_{i\eta} - \int_{\partial_\infty^{reg}} U_{\eta\xi} \mathbb{P}_i\{X_\zeta \in d\xi\}}{w_0 - \int_{\partial_\infty} U_{\eta\xi} \mathbb{P}_r\{X_\zeta \in d\xi\}} = \frac{U_{i\eta} - \int_{\partial_\infty^{reg}} U_{\eta\xi} \kappa(i, \xi) \mathbb{P}_r\{X_\zeta \in d\xi\}}{w_0 - \int_{\partial_\infty} U_{\eta\xi} \mathbb{P}_r\{X_\zeta \in d\xi\}}.$$

## 4 Trees Potential without Absorption

### 4.1 Reflecting at the root

Let  $(I, \mathcal{T})$  be a tree rooted at  $r$ . In this section we consider the case when  $r$  is a reflecting barrier. As before we take a strictly positive and strictly increasing sequence  $(w_n : n \in \mathbb{N})$  and consider a symmetric  $q$ -matrix  $Q$  on  $I \times I$ , supported on the tree and the diagonal, defined as in (2.2) except at the pair  $(r, r)$ , where  $Q_{rr} = -\frac{|S_r|}{w_1 - w_0}$ . It is direct to check that  $Q$  is conservative:  $\sum_{j \in I} Q_{ij} = 0$  for every  $i \in I$ . We assume the Markov process  $(X_t)$  associated to  $Q$  is transient, that is  $\mathbb{P}_r\{X_\zeta \in \partial_\infty\} = 1$ , and that all points in  $\partial_\infty$  are regular.

The aim is to obtain a representation of the potential  $V$  for this process as well as for the Martin kernel, in terms of the tree matrix  $U = (U_{ij} = w_{|i \wedge j|} : i, j \in I)$ . For this purpose, consider the translated matrix  $U^{(a)} := U + a$ , for  $a > 0$ , which is the tree matrix

associated to the level function  $w_n^{(a)} = w_n + a$ . Define the matrix  $Q^{(a)}$  on  $I \times I$  as in (2.2) with respect to this level function. At  $(r, r)$  it takes the value  $Q_{rr}^{(a)} = Q_{rr} - \frac{1}{w_0+a}$ . We also put  $Q_{r\partial_r}^{(a)} = \frac{1}{w_0+a}$ . We notice that the matrices  $Q^{(a)}$  and  $Q$  in  $I \times I$ , only differ at  $(r, r)$ .

As  $a$  tends to infinity,  $Q^{(a)}$  converges to  $Q$ , and the associated processes also converge. In fact, a coupling argument allows us to construct an increasing sequence of stopping times  $T^{(a)} \uparrow_{a \rightarrow \infty} \infty$  such that

$$X_t^{(a)} = X_t \text{ if } t < T^{(a)} \text{ and } X_t^{(a)} = \partial_r \text{ if } t \geq T^{(a)},$$

is a Markov process with generator  $Q^{(a)}$ . Notice that the lifetime variables  $\zeta^{(a)}$  and  $\zeta$  associated respectively to  $X$  and  $X^{(a)}$ , verify  $\zeta^{(a)} = \zeta \wedge T^{(a)}$ . From this representation it also follows immediately that the potentials  $V^{(a)}$  and  $V$ , associated to  $Q^{(a)}$  and  $Q$ , respectively, verify  $\forall i, j, V_{ij}^{(a)} \uparrow_{a \rightarrow \infty} V_{ij}$ . Therefore, the representation (3.2) reads as follows

$$U_{ij}^{(a)} - V_{ij}^{(a)} = \int_{\partial_\infty} U_{\eta j}^{(a)} \mathbb{P}_i\{X_\zeta \in d\eta, \zeta \leq T^{(a)}\},$$

or equivalently

$$U_{ij} - V_{ij}^{(a)} = \int_{\partial_\infty} U_{\eta j} \mathbb{P}_i\{X_\zeta \in d\eta, \zeta \leq T^{(a)}\} - a\mathbb{P}_i\{T^{(a)} < \zeta\}.$$

Passing to the limit  $a \rightarrow \infty$  we obtain that  $\lim_{a \rightarrow \infty} a\mathbb{P}_i\{T^{(a)} < \zeta\}$  exists and moreover

$$U_{ij} - V_{ij} = \int_{\partial_\infty} U_{\eta j} \mathbb{P}_i\{X_\zeta \in d\eta\} - \lim_{a \rightarrow \infty} a\mathbb{P}_i\{T^{(a)} < \zeta\}.$$

Substituting  $j$  by  $r$  in the last equality and using that  $U_{ir} = U_{\eta r} = w_0$ , we find  $\lim_{a \rightarrow \infty} a\mathbb{P}_i\{T^{(a)} < \zeta\} = V_{ir}$ , and therefore we get

$$U_{ij} - V_{ij} = \int_{\partial_\infty} U_{\eta j} \mathbb{P}_i\{X_\zeta \in d\eta\} - V_{ir}. \quad (4.1)$$

Now, if we take  $j \rightarrow \xi \in \partial_\infty^{reg}$  we obtain

$$V_{ir} = \int_{\partial_\infty} U_{\eta \xi} \mathbb{P}_i\{X_\zeta \in d\eta\} - U_{i\xi} = \int_{\partial_\infty} (U_{\eta \xi} - U_{i\xi}) \mathbb{P}_i\{X_\zeta \in d\eta\}.$$

Thus, we have proven that the following equality holds

$$U_{ij} - V_{ij} = \int_{\partial_\infty} (U_{\eta j} + U_{i\xi} - U_{\eta \xi}) \mathbb{P}_i\{X_\zeta \in d\eta\}, \quad (4.2)$$

which is independent of  $\xi \in \partial_\infty^{reg}$ . Integrating (4.2) with respect to  $\mathbb{P}_j\{X_\zeta \in d\xi\}$  gives

$$\begin{aligned} U_{ij} - V_{ij} &= \int_{\partial_\infty} U_{\eta j} \mathbb{P}_i\{X_\zeta \in d\eta\} + \int_{\partial_\infty} U_{i\xi} \mathbb{P}_j\{X_\zeta \in d\xi\} - \\ &\quad \int_{\partial_\infty} \int_{\partial_\infty} U_{\eta \xi} \mathbb{P}_j\{X_\zeta \in d\xi\} \mathbb{P}_i\{X_\zeta \in d\eta\}. \end{aligned}$$

The Martin kernel  $\kappa^{(a)}$  associated to  $Q^{(a)}$  can be computed as in Theorem 3.3. Take  $i \in I$ ,  $\eta \in \partial_\infty$  and  $n > |i \wedge \eta|$  then

$$\kappa^{(a)}(i, \eta) = \frac{\mathbb{P}_i\{X_\zeta \in C^n(\eta), \zeta \leq T^{(a)}\}}{\mathbb{P}_r\{X_\zeta \in C^n(\eta), \zeta \leq T^{(a)}\}}.$$

Therefore, there is also continuity of the Martin kernel with respect to  $a$ . Passing to the limit  $a \rightarrow \infty$  and using the representation (3.29) we obtain

$$\kappa(i, \eta) = \lim_{a \rightarrow \infty} \sum_{k=0}^{|i \wedge \eta|+1} \frac{1}{G_k^{(a)}} (\mathbb{E}_{\mu^{(a)}}(U_{i\bullet}^{(a)} | \mathcal{F}_k)(\eta) - \mathbb{E}_{\mu^{(a)}}(U_{i\bullet}^{(a)} | \mathcal{F}_{k-1})(\eta)), \quad (4.3)$$

where

$$\begin{aligned} G_k^{(a)} &= G_k^{(a)}(\eta) = \sum_{n \geq k} (w_n^{(a)} - w_{n-1}^{(a)}) \mu^{(a)}(C^n(\eta)), \\ \mu^{(a)}(\bullet) &= \frac{\mathbb{P}_r\{X_\zeta \in \bullet, \zeta \leq T^{(a)}\}}{\mathbb{P}_r\{\zeta \leq T^{(a)}\}}. \end{aligned} \quad (4.4)$$

We notice

$$G_0^{(a)} = \frac{w_0 + a}{\mathbb{P}_r\{\zeta \leq T^{(a)}\}},$$

and

$$\begin{aligned} G_k^{(a)} &= G_0^{(a)} - \sum_{n=0}^{k-1} (w_n^{(a)} - w_{n-1}^{(a)}) \mu^{(a)}(C^n(\eta)) \\ &= \frac{w_0 + a}{\mathbb{P}_r\{\zeta \leq T^{(a)}\}} - (w_0 + a) - \sum_{n=1}^{k-1} (w_n - w_{n-1}) \mu^{(a)}(C^n(\eta)) \\ &= (w_0 + a) \frac{\mathbb{P}_r\{T^{(a)} < \zeta\}}{\mathbb{P}_r\{\zeta \leq T^{(a)}\}} - \sum_{n=1}^{k-1} (w_n - w_{n-1}) \mu^{(a)}(C^n(\eta)). \end{aligned}$$

Therefore, the previous computations show the following result.

**Theorem 4.1** *Let  $\mu(\bullet) = \mathbb{P}_r\{X_\zeta \in \bullet\}$ . Consider  $G_0(\eta) := \int U_{\eta\xi} \mathbb{P}_r\{X_\zeta \in d\xi\}$  and  $G_k := \lim_{a \rightarrow \infty} G_k^{(a)}$ . Then  $G_0(\eta) = V_{rr} + w_0$  is a constant and  $(G_k : k \geq 1)$  is a positive decreasing predictable process that verifies*

$$G_k(\eta) = G_0 - \sum_{n=0}^{k-1} \Delta_n(w) \mu(C^n(\eta)) = \sum_{n \geq k} \Delta_n(w) \mu(C^n(\eta)) \text{ for } k \geq 1;$$

and the following representation holds

$$\kappa(i, \eta) = 1 + \sum_{k=1}^{|i \wedge \eta|+1} \frac{1}{G_k(\eta)} (\mathbb{E}_\mu(U_{i\bullet} | \mathcal{F}_k)(\eta) - \mathbb{E}_\mu(U_{i\bullet} | \mathcal{F}_{k-1})(\eta)). \quad (4.5)$$

**Remark 4.1** *It can be shown that  $\mu^{(a)}$  defined in (4.4) does not depend on  $a \geq 0$  (recall that  $\mu^{(0)}$  is the measure defined in (3.5) in subsection 3.1 for the chain absorbed at  $\partial_r$ ). Indeed this follows from the independence relation*

$$\mathbb{P}_r\{X_\zeta \in \bullet, \zeta \leq T^{(a)}\} = \mathbb{P}_r\{X_\zeta \in \bullet\} \mathbb{P}_r\{\zeta \leq T^{(a)}\},$$

then

$$\mu^{(a)} = \mu \text{ for } a \geq 0, \quad \text{where } \mu(\bullet) = \mathbb{P}_r\{X_\zeta \in \bullet\}.$$

Further, if  $N_r^*$  is the number of visits in the strict future to  $r$  of the discrete skeleton of  $(X_t)$ , then a simple argument shows that  $\mu(\bullet) = \mathbb{P}_r\{X_\zeta \in \bullet | N_r^* = 0\}$ .

**Remark 4.2** *If we take*

$$W^{(a)} = \sum_{n \in \mathbb{N}} G_n^{(a)} (\mathbb{E}_{\mu^{(a)}}(\cdot | \mathcal{F}_n) - \mathbb{E}_{\mu^{(a)}}(\cdot | \mathcal{F}_{n-1})),$$

then

$$\lim_{a \rightarrow \infty} (W^{(a)})^{-1} = \sum_{n \geq 1} G_n^{-1} (\mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1})).$$

coincides with the operator  $\underline{W}^{-1} := W^{-1} - G_0^{-1} \mathbb{E}_\mu$  defined in (3.19) in Remark 3.4, it verifies  $\lim_{a \rightarrow \infty} (W^{(a)})^{-1} \mathbf{1} = 0$  and it is the generator of a Markov process defined in the boundary  $\partial_\infty$  that will be studied in section 6.

## 4.2 Potential for Homogeneous Trees

In this section we consider standard random walk on a homogeneous tree of degree  $p+1 \geq 3$  and we show that in this case the previous calculations give a close form to the Martin kernel. Some of these computations are well known, see for instance [39]. We assume  $\mathcal{T}$  is an infinite rooted tree, with  $|S_r| = p+1$  and  $|S_i| = p$  for  $i \neq r$ . As a weight function we take  $w_n = n+1$ . Finally, we assume that  $r$  is reflecting. In this way we have

$$Q_{ii^+} = 1, \quad Q_{ii} = -(p+1) \text{ for } i \in I \text{ and } Q_{ii^-} = 1 \text{ for } i \neq r.$$

It is well known that this tree matrix is transient for all  $p \geq 2$ .

From symmetry considerations  $\mu$  is the uniform measure on  $\partial_\infty$  and it is easy to see that all points in  $\partial_\infty$  are regular. Let us now compute the quantities involved on (4.5).

We fix  $i \in I$ ,  $\eta \in \partial_\infty$  and put  $n = |i \wedge \eta|$ ,  $|i| = m$ . We assume  $m \geq 1$  because for  $m = 0$  we have  $i = r$  and  $\kappa(r, \eta) = 1$ . We set  $C^k = C^k(\eta) = [\eta(k), \infty] \cap \partial_\infty$ . Therefore,  $\mu(C^k) = ((p+1)p^{k-1})^{-1}$  for all  $k \geq 1$ , and  $\mu(C^0) = 1$ . Then,

$$G_k(\eta) = \sum_{l \geq k} (w_l - w_{l-1}) \mu(C^l(\eta)) = \sum_{l \geq k} \frac{1}{(p+1)p^{l-1}} = \frac{1}{(p^2-1)p^{k-2}} \text{ for } k \geq 1.$$

We need to compute  $\mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_k)(\eta)$  when  $k \leq n+1$ . By definition we have

$$\mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_k)(\eta) = \frac{1}{\mu(C^k)} \int_{C^k} U_{i\xi} \mu(d\xi) = \begin{cases} \frac{1}{(p+1)p^{m-1}} \sum_{\substack{j \in I \\ |j|=m}} U_{ij} & \text{if } k = 0; \\ \frac{(p+1)p^{k-1}}{(p+1)p^{m-1}} \sum_{\substack{j \in C^k \\ |j|=m}} U_{ij} & \text{if } 1 \leq k \leq n+1. \end{cases}$$

If  $k = n+1$  this gives  $\mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_{n+1})(\eta) = n+1$ . When  $0 \leq k \leq n$ , the values of  $\{U_{ij} : j \in C^k, |j| = m\}$  range from  $k+1$  to  $m+1$ . For a given integer  $t$  in this range denote by  $M_t^k$  the number of sites  $j$  for which  $U_{ij} = t$ . We have  $M_{m+1}^k = 1$  and

$$M_m^k = p-1, M_{m-1}^k = (p-1)p, \dots, M_t^k = (p-1)p^{m-t}, \dots, M_{k+1}^k = (p-1)p^{m-(k+1)} \text{ for } k \geq 1;$$

$$M_m^0 = p-1, M_{m-1}^0 = (p-1)p, \dots, M_t^0 = (p-1)p^{m-t}, \dots, M_2^0 = (p-1)p^{m-2}, M_1^0 = p^m \text{ for } k = 0.$$

From these expressions we obtain

$$\begin{aligned} \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_0)(\eta) &= \frac{1}{(p+1)p^{m-1}} \left( m+1 + p^m + (p-1) \sum_{t=2}^m tp^{m-t} \right), \\ \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_k)(\eta) &= p^{k-m} \left( m+1 + (p-1) \sum_{t=k+1}^m tp^{m-t} \right) \text{ for } 1 \leq k \leq n. \end{aligned}$$

Hence we get

$$\begin{aligned} \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_1)(\eta) - \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_0)(\eta) &= p^2 \frac{1 - p^{-m}}{p^2 - 1} \\ \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_k)(\eta) - \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_{k-1})(\eta) &= 1 - p^{k-m-1} \text{ for } 2 \leq k \leq n; \\ \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_{n+1})(\eta) - \mathbb{E}_\mu(U_{i\bullet}|\mathcal{F}_n)(\eta) &= \frac{p^{n-m} - 1}{p-1}. \end{aligned}$$

Finally we obtain (see for example [39] Theorem 8.1)

$$\kappa(i, \eta) = p^{2n-m}, \text{ where } m = |i|, n = |i \wedge \eta|.$$

In particular if  $|i \wedge \eta| = 0$  we get for  $k \geq 1$

$$p^{-m} = \kappa(i, \eta) = \frac{\mathbb{P}_i\{X_\zeta \in C^k(\eta)\}}{\mathbb{P}_r\{X_\zeta \in C^k(\eta)\}} = \mathbb{P}_i\{T_r < \infty\}.$$

In a similar way we obtain  $\mathbb{P}_i\{T_{i-} < \infty\} = p^{-1}$ . From (4.1) we have

$$V_{rj} - V_{rr} = V_{rr}(\mathbb{P}_j\{T_r < \infty\} - 1) = 1 - \int U_{j\eta} \mathbb{P}_r\{X_\zeta \in d\eta\},$$

where for the last integral we assume  $|j| = m \geq 1$ , and we get

$$\int U_{j\eta} \mathbb{P}_r\{X_\zeta \in d\eta\} = \frac{1}{(p+1)p^{m-1}} \sum_{t=1}^{m+1} t M_t^0.$$

From this expression we find

$$V_{rr} = \frac{p}{(p+1)(p-1)} \quad \text{and} \quad V_{jr} = \frac{p^{1-m}}{(p+1)(p-1)}.$$

A simple argument based on time reversal shows that  $\mathbb{P}_r\{T_j < \infty\} = \mathbb{P}_j\{T_r < \infty\} = p^{-|j|}$ , and in general

$$\mathbb{P}_i\{T_j < \infty\} = p^{-|\text{geod}(i,j)|}.$$

Since  $V_{jr} = \mathbb{P}_r\{T_j < \infty\} V_{jj}$  we deduce that

$$V_{jj} = \frac{p}{(p+1)(p-1)},$$

which can be also obtained from the invariance of the tree under translations. Using the same argument, if  $|\text{geod}(i,j)| = m = |k|$  we have

$$V_{ij} = V_{kr} = \frac{p^{1-m}}{(p+1)(p-1)} = \frac{p^{1-|\text{geod}(i,j)|}}{(p+1)(p-1)}.$$

Finally, from (4.1) we get that

$$\int U_{j\eta} \mathbb{P}_i\{X_\zeta \in d\eta\} = |i \wedge j| + 1 + p \frac{p^{-|i|} - p^{-|\text{geod}(i,j)|}}{(p+1)(p-1)}.$$

## 5 Ultrametricity

There is a wide literature concerning ultrametricity, but it is not a common notion in potential theory. So, we supply some basic properties following from the ultrametric inequality (in our notation, the ultrametric inequality is the one verified by  $1/d$ , being  $d$  an ultrametric distance). The core of this section are subsections 5.3 and 5.4, where the Markov semigroup and the harmonic functions emerging from the ultrametric matrix, in terms of the minimal tree matrix extension are constructed.

### 5.1 Basic Notions and the Minimal Rooted Tree Extension

We impose conditions in order that an ultrametric matrix can be immersed in a countable and locally finite tree. It is known that a tree structure is behind an ultrametric (for a



deep study of this relation see [22]), but we prefer here to give an explicit construction because it allows a better understanding of the main results of this section.

We note that up to Lemma 5.2 the set  $I$  will have no restriction. Most of the properties we present are easily deduced from the ultrametric inequality, so they are established without a proof.

**Definition 5.1**  $U = (U_{ij} : i, j \in I)$  is an ultrametric arrangement if it is symmetric, that is  $U_{ij} = U_{ji}$  for any couple  $i, j \in I$ , and verifies the ultrametric inequality

$$U_{ij} \geq \min\{U_{ik}, U_{kj}\} \text{ for any } i, j, k \in I.$$

In particular  $U_{ii} \geq U_{ij}$  for any  $i, j \in I$ , so  $U_{ij} = U_{ii} \Rightarrow U_{jj} \geq U_{ii}$ .

Observe that for any triple  $i_1, i_2, i_3 \in I$  there exists a permutation  $\varphi$  of  $\{1, 2, 3\}$  such that

$$U_{i_{\varphi(1)}i_{\varphi(2)}} = \min\{U_{i_{\varphi(2)}i_{\varphi(3)}}, U_{i_{\varphi(3)}i_{\varphi(1)}}\}.$$

Hence,  $U_{ik} > U_{kj} \Rightarrow U_{ik} > U_{kj} = U_{ij}$  and  $U_{ik} = U_{kj} \Rightarrow U_{ij} \geq U_{ik} = U_{kj}$ .

Let us introduce the equivalence relation

$$i \sim j \Leftrightarrow (\forall k \in I : U_{ik} = U_{jk}).$$

Notice that  $i \sim j \Leftrightarrow U_{ii} = U_{ij} = U_{jj}$ . Let us introduce the relation

$$i \preceq j \Leftrightarrow U_{ij} = U_{ii}. \quad (5.1)$$

From  $U_{jk} \geq \min\{U_{ji}, U_{ik}\} = \min\{U_{ii}, U_{ik}\} = U_{ik}$ . we get

$$i \preceq j \Leftrightarrow U_{i\bullet} \leq U_{j\bullet} \text{ ( that is } \forall k \in I : U_{ik} \leq U_{jk}),$$

so the relation  $\preceq$  is a preorder, that is it is reflexive and transitive. The equivalence relation associated to the preorder  $\preceq$  is  $\sim$ , this means  $[i \preceq j \text{ and } j \preceq i] \Leftrightarrow i \sim j$ . On the other hand  $i \preceq j \Rightarrow U_{ii} \leq U_{jj}$ .

Now, we denote  $i\mathcal{G}j$  if  $i, j$  are comparable, that is  $i \preceq j$  or  $j \preceq i$ . We have  $i\mathcal{G}j \Leftrightarrow U_{ij} \geq \min\{U_{ii}, U_{jj}\}$ . From definition we also get  $i \sim j \Leftrightarrow [U_{ii} = U_{jj} \text{ and } i\mathcal{G}j]$ . The left and the right intervals defined by  $i \in I$  are respectively

$$[i, \infty)^U = \{j \in I : i \preceq j\} \text{ and } (-\infty, i]^U = \{j \in I : j \preceq i\}.$$

Notice that for any  $i \in I$  the set  $(-\infty, i]^U$  is  $\preceq$ -totally preordered. This means that for

$$\forall j, k \in (-\infty, i]^U \text{ we have } j\mathcal{G}k.$$

Some elementary properties deduced from the ultrametric hypothesis are summarized below, they are easily proven by analysis of cases. The first two relations reveal a hierarchical structure.

- (i)  $[i\mathcal{G}j, k \in [i, \infty)^U, \ell \in [j, \infty)^U]$  implies  $k\mathcal{G}\ell$ .
- (ii) If  $i\mathcal{G}j$  then  $[i, \infty)^U \cap [j, \infty)^U = \emptyset$ .
- (iii)  $i \preceq j$  and  $k \preceq \ell$  imply  $U_{j\ell} \geq U_{ik}$ .
- (iv)  $i \preceq j$  implies that for any  $k \in I$  it holds ( $i \preceq k$  or  $U_{jk} = U_{ik}$ ).
- (v)  $i \preceq j$  and  $k \preceq \ell$  imply ( $i\mathcal{G}k$  or  $U_{j\ell} = U_{ik}$ ).

In the sequel we will assume the following condition holds

$$i \sim j \Leftrightarrow i = j. \quad (H1)$$

Property (H1) is equivalent to the fact that  $\preceq$  is an order, or equivalently to the relation  $i \neq j \Rightarrow U_{ij} < \max\{U_{ii}, U_{jj}\}$ . We point out that if  $I$  is finite and  $U > 0$  condition (H1) is equivalent to the nonsingularity of  $U$  (see [15], [33] or [36]).

We denote  $\mathcal{W} = \{U_{ij} : i, j \in I\}$  the set of values of  $U$ . To every  $w \in \mathcal{W}$  we associate the nonempty set  $J(w) = \{i \in I : U_{ii} \geq w\}$  and the relation

$$i \equiv_w j \Leftrightarrow U_{ij} \geq w.$$

The ultrametric inequality implies that  $\equiv_w$  is an equivalence relation in  $J(w)$ . By  $E^w$  we mean an equivalence class of  $\equiv_w$ , and  $E_i^w$  denotes the equivalence class containing  $i \in J(w)$ . In the case  $U_{ii} < w$ , that is  $i \notin J(w)$ , we put  $E_i^w = \emptyset$ . As usual  $J(w)/\equiv_w$  denotes the set of equivalence classes of elements of  $J(w)$ .

Let us introduce the following set

$$\tilde{I} = \{(E^w, w) : E^w \in J(w)/\equiv_w, w \in W\}.$$

The function

$$\mathbf{i}^U : I \rightarrow \tilde{I}, \mathbf{i}^U(i) = (E_i^{U_{ii}}, U_{ii})$$

is one-to-one. In fact, if  $\mathbf{i}^U(i) = \mathbf{i}^U(j)$ , then  $U_{ij} \geq U_{ii} = U_{jj}$ . From condition (H1) we deduce  $i = j$ . In this way we identify  $i \in I$  with  $\mathbf{i}^U(i) = (E_i^{U_{ii}}, U_{ii}) \in \tilde{I}$ .

Observe that  $E_i^{U_{ii}} = [i, \infty)^U$ , for every  $i \in I$ . Also it holds

$$\left[ w \leq w' \Rightarrow E^{w'} \subseteq E^w \right] \text{ and } \left[ (w \leq w', E^{w'} \neq E^w) \Rightarrow w < w' \right].$$

**Lemma 5.1** *If  $E^{w'} \not\subseteq E^w$  and  $E^w \not\subseteq E^{w'}$ , then*

$$\forall k, k' \in E^w, \forall \ell, \ell' \in E^{w'} : U_{k\ell} = U_{k'\ell'} < \min\{w, w'\} \text{ and } E^w \cap E^{w'} = \emptyset.$$

**Proof.** Let  $k \in E^w \setminus E^{w'}$  and  $l \in E^{w'} \setminus E^w$ . Also take  $k' \in E^w, l' \in E^{w'}$ . Since  $\equiv_w, \equiv_{w'}$  are equivalent relations on their respective domains, we get  $U_{kl'} < w'$  and  $U_{k'l} < w$ . In particular  $U_{kl} < \min\{w, w'\}$ . On the other hand the definition of  $E^w$  implies  $U_{kk'} \geq w$ . Using the ultrametric property we get  $U_{k'l} \geq \min\{U_{k'k}, U_{kl}\} = U_{kl}$ , and similarly  $U_{kl} \geq$

$U_{k'l}$ , from which the equality  $U_{kl} = U_{k'l}$  holds. In an analogous way it is deduced the equality  $U_{kl} = U_{kl'}$ , and we get that

$$k' \in E^w \setminus E^{w'} \text{ and } l' \in E^{w'} \setminus E^w.$$

This implies that  $E^w \cap E^{w'} = \emptyset$ . Again using the ultrametricity we find

$$U_{k'l'} \geq \min\{U_{k'l}, U_{ll'}\} = U_{k'l} = U_{kl}.$$

By exchanging the roles of  $k$  with  $k'$  and  $l$  with  $l'$ , we deduce the result.  $\square$

The previous result implies that two classes  $E^w$  and  $E^{w'}$  are either disjoint or one is included in the other. Now we define  $\tilde{U}$ , an extension of  $U$  to  $\tilde{I}$ .

**Definition 5.2** Let  $\tilde{i} = (E^w, w) \in \tilde{I}$ ,  $\tilde{j} = (E^{w'}, w') \in \tilde{I}$ . If  $E^{w'} \subseteq E^w$  or  $E^w \subseteq E^{w'}$  we put  $\tilde{U}_{\tilde{i}\tilde{j}} = \min\{w, w'\}$ . On the contrary, that is  $E^w \cap E^{w'} = \emptyset$ , we put  $\tilde{U}_{\tilde{i}\tilde{j}} = U_{k\ell}$ , where  $k \in E^w$  and  $\ell \in E^{w'}$ .

From Lemma 5.1,  $\tilde{U}$  is well defined. On the other hand it is direct to prove that for any  $i, j \in I$  it holds  $U_{ij} = \tilde{U}_{i^v(j)}$ . In this way  $\tilde{U}$  is an extension of  $U$ . Also, if  $i \in E^w, j \in E^{w'}$  then  $U_{ij} \geq \tilde{U}_{\alpha\beta}$ , where  $\alpha = (E^w, w)$ ,  $\beta = (E^{w'}, w')$ .

**Lemma 5.2**  $\tilde{U} = (\tilde{U}_{\tilde{i}\tilde{j}} : \tilde{i}, \tilde{j} \in \tilde{I})$  is ultrametric.

**Proof.** For  $u, v, w \in W$  consider the following elements of  $\tilde{I}$ :  $\tilde{i} = (E^u, u)$ ,  $\tilde{j} = (E^v, v)$  and  $\tilde{k} = (E^w, w)$ . Take  $i \in E^u, j \in E^v, k \in E^w$ . The proof is divided into two cases.

**Case 1.** We assume  $E^u \cap E^v = \emptyset$ . The ultrametric property satisfied by  $U$  and the definition of  $\tilde{U}$  imply  $\tilde{U}_{\tilde{i}\tilde{j}} = U_{ij} \geq \min\{U_{ik}, U_{kj}\} \geq \min\{\tilde{U}_{\tilde{i}\tilde{k}}, \tilde{U}_{\tilde{k}\tilde{j}}\}$ . Then the property holds.

**Case 2.** We assume, without lost of generality that  $E^u \subseteq E^v$  and  $v \leq u$ . If  $E^w \cap E^v = \emptyset$  one gets that  $\tilde{U}_{\tilde{j}\tilde{k}} = U_{jk} < v = \tilde{U}_{\tilde{i}\tilde{j}}$  and the property is verified. Finally, if  $E^w \cap E^v \neq \emptyset$  then  $\tilde{U}_{\tilde{j}\tilde{k}} = \min\{v, w\} \leq v = \tilde{U}_{\tilde{i}\tilde{j}}$ .  $\square$

In the sequel we shall assume  $I$  is countable and the following hypothesis holds

$$\mathcal{W} = \{U_{ij} : i, j \in I\} \subset \mathbb{R}_+^* \text{ has no finite accumulation point.} \quad (H2)$$

We put  $\mathcal{W} = \{w_n : n \in \mathbb{N}\}$  where  $(w_n)$  increases with  $n \in \mathbb{N}$ ,  $w_0 > 0$ . Under (H2) we are able to define in  $\tilde{I}$  the following binary relation  $\tilde{\mathcal{T}}$ . For  $u, v \in \mathcal{W}$  we set

$$((E^u, u), (E^v, v)) \in \tilde{\mathcal{T}} \Leftrightarrow \exists n \in \mathbb{N} : \{u, v\} = \{w_n, w_{n+1}\} \text{ and } E^u \cap E^v \neq \emptyset.$$

Two points  $\tilde{i}, \tilde{j} \in \tilde{I}$  are said to be neighbors in  $\tilde{\mathcal{T}}$  if  $(\tilde{i}, \tilde{j}) \in \tilde{\mathcal{T}}$ .

Observe that if  $((E^{w_n}, w_n), (E^{w_{n+1}}, w_{n+1})) \in \tilde{\mathcal{T}}$ , then  $E^{w_{n+1}} \subseteq E^{w_n}$ . The strict inclusion  $E^{w_{n+1}} \subsetneq E^{w_n}$  holds if and only if there exists a unique  $i \in E^{w_n}$  such that  $w_n = U_{ii}$ . Indeed, it suffices to show the uniqueness. Let  $i \in E^{w_n} \setminus E^{w_{n+1}}$  then  $w_n \leq U_{ii} < w_{n+1}$ . For any other  $k \in E^{w_n}$  for which  $U_{kk} = w_n$  it holds  $U_{ik} \geq w_n$ . We get  $i \sim k$  and from (H1) we conclude  $i = k$ .

It is easy to see that  $(\tilde{I}, \tilde{\mathcal{T}})$  is a tree rooted at  $\tilde{r}$ , where  $\tilde{U}_{\tilde{r}\tilde{r}} = w_0$ . This point  $\tilde{r}$  exists (and it is unique) because either there exists  $i_0 \in I$  verifying  $U_{i_0 i_0} = w_0$  in which case  $\tilde{r} = \tilde{i}_0$ , or in the contrary, our construction adds a point  $\tilde{r} \in \tilde{I} \setminus I$  such that  $\tilde{U}_{\tilde{r}\tilde{r}} = w_0$ .

By construction  $\tilde{U}$  is the minimal tree matrix extending  $U$ , that is we can immerse  $\tilde{U}$  in any other tree extension of  $U$ . The tree  $(\tilde{I}, \tilde{\mathcal{T}})$ , supporting this minimal extension, is locally finite if and only if the following assumption is verified

$$\forall w \in \mathcal{W} \text{ it holds } |J(w)/\equiv_w| < \infty. \quad (H3)$$

Since  $(\tilde{I}, \tilde{\mathcal{T}})$  is a rooted tree, all the concepts defined in the Introduction applied to it. In particular we denote by  $\tilde{\preceq}$  the order relation introduced in (1.1); by  $\tilde{\wedge}$  the associated minimum, by  $[\tilde{i}, \infty)$  the branch born at  $\tilde{i}$  and by  $geod(\tilde{i}, \tilde{j})$  the geodesic between two points in  $\tilde{I}$ . Since we have identified  $i \in I$  with  $\mathbf{i}^U(i) \in \tilde{I}$ , all these concepts have a meaning for elements in  $I$ . In particular  $\tilde{\preceq}$  is an extension of the order relation  $\preceq$  defined on  $I$  on (5.1), and we have the equality  $[i, \infty)^U = [i, \infty) \cap I$ .

Observe that the  $\tilde{\preceq}$ -minimum in  $(\tilde{I}, \tilde{\mathcal{T}})$  is characterized as follows. Take  $(E^u, u), (E^v, v) \in \tilde{I}$ , and any  $i \in E^u$ , then  $(E^u, u) \tilde{\wedge} (E^v, v) = E_i^w$ , where  $w = \sup\{z \in \mathcal{W} : z \leq u, E_i^z \supseteq E^v\}$ . Notice that  $E_i^{w_0} = I$ .

## 5.2 Neighbor Relation

We will assume that hypotheses (H1)-(H3) are fulfilled. The next definition is a notion of neighbor on  $I$  giving a better understanding of the embedding  $I$  in  $\tilde{I}$ , in particular to describe how the elements in  $\tilde{I} \setminus I$  are surrounded by  $I$ .

**Definition 5.3** *Let  $i \in I$ .*

(i) *The set  $\mathcal{V}(i) = \{j \in I : j \neq i, geod(i, j) \cap I = \{i, j\}\}$  is called the set of  $U$ -neighbors of  $i$ . We will also put  $\mathcal{V}^*(i) = \mathcal{V}(i) \cup \{i\}$ .*

(ii) *The set  $\mathcal{B}(i) = \{\tilde{j} \in \tilde{I} : geod(i, \tilde{j}) \cap I \subseteq \{i, \tilde{j}\}\}$  is called the attraction basin of  $i$ .*

Notice that  $\mathcal{V}^*(i) \subseteq \mathcal{B}(i)$ . In the next result we summarize some useful properties of  $\mathcal{B}(i)$ ,  $\mathcal{V}(i)$  and  $\mathcal{V}^*(i)$ .

**Lemma 5.3** (i)  $\tilde{j} \in \mathcal{B}(i) \setminus \mathcal{V}(i)$  if and only if  $\text{geod}(\tilde{j}, i) \cap I = \{i\}$ . Moreover  $\mathcal{V}^*(i) = \mathcal{B}(i) \cap I$  and  $\mathcal{B}(i) \setminus \mathcal{V}^*(i) = \mathcal{B}(i) \setminus I$ .

(ii) If  $\tilde{j} \in \mathcal{B}(i) \setminus \mathcal{V}^*(i)$  then all its neighbors in  $(\tilde{I}, \tilde{\mathcal{T}})$  belong to  $\mathcal{B}(i)$ . Thus,  $(\mathcal{B}(i), \tilde{\mathcal{T}}|_{\mathcal{B}(i) \times \mathcal{B}(i)})$  is a tree. If we fix the root of this tree at  $i$  then the set of leaves is  $\mathcal{V}(i)$ .

(iii) For every  $\tilde{j} \notin \mathcal{B}(i)$  there exists a unique  $k = k(i) \in \mathcal{V}(i)$  such that  $\text{geod}(i, \tilde{j}) \cap \mathcal{V}^*(i) = \{i, k\}$ . This unique  $k$  also verifies that  $k \in \text{geod}(\tilde{l}, \tilde{j}) \cap \mathcal{V}^*(i)$  for every  $\tilde{l} \in \mathcal{B}(i)$ .

(iv) For every  $\tilde{j} \in \tilde{I}$  there exists  $i \in I$  such that  $\tilde{j} \in \mathcal{B}(i)$ .

(v) For  $j \in \mathcal{V}(i)$  either  $(i, j) \in \tilde{\mathcal{T}}$ , that is  $i, j$  are neighbors on  $\tilde{\mathcal{T}}$ , or there is a unique  $\tilde{k} \in \tilde{I} \setminus I$  such that  $(\tilde{k}, i) \in \tilde{\mathcal{T}}$  and  $\tilde{k} \in \mathcal{B}(j) \cap \text{geod}(i, j)$ .

**Proof.**

(i) and (ii) are direct from the definitions.

(iii) Take  $\tilde{j} \notin \mathcal{B}(i)$ . If  $\text{geod}(\tilde{j}, i) \cap \mathcal{V}^*(i) = \{i\}$  then  $\text{geod}(\tilde{j}, i) \cap I = \{i\}$ . In fact, if this intersection contains another point  $\ell \in I$  and if we take  $m \in (\text{geod}(\ell, i) \cap I) \setminus \{i\}$ , the closest point to  $i$ , we obtain  $m \in \mathcal{V}^*(i)$  which is a contradiction. Therefore,  $\text{geod}(\tilde{j}, i) \cap I = \{i\}$  and then  $\tilde{j} \in \mathcal{B}(i)$  which is also a contradiction.

Thus we can assume  $|\text{geod}(\tilde{j}, i) \cap \mathcal{V}^*(i)| \geq 2$ . If this intersection has at least 3 points, from the inclusion  $\text{geod}(\tilde{j}, i) \subseteq \text{geod}(\tilde{\ell}, \tilde{j}) \cup \text{geod}(\tilde{\ell}, i)$  for any  $\tilde{\ell} \in \tilde{I}$ , we would find a point  $k \in \mathcal{V}^*(i)$  for which  $\text{geod}(k, i) \cap I$  contains at least 3 points. This is a contradiction, and the result follows.

(iv) For  $\tilde{j}$  and  $k \in I$  we consider  $\text{geod}(\tilde{j}, k)$ . The first point in this geodesics (when starting from  $\tilde{j}$ ) belonging to  $I$  makes the job.

(v) If  $i, j$  are not neighbors in  $\tilde{\mathcal{T}}$  then  $\text{geod}(i, j)$  contains strictly  $\{i, j\}$ . Take  $\tilde{k} \neq i$  the closest point to  $i$  in  $\text{geod}(i, j)$ . Clearly  $\tilde{k} \in \tilde{I} \setminus I$ , otherwise  $j \notin \mathcal{V}^*(i)$ . By the same reason  $\text{geod}(\tilde{k}, j) \cap I = \{j\}$  and therefore  $\tilde{k} \in \mathcal{B}(j)$ .  $\square$

Let us fix some  $\tilde{j} \in \tilde{I} \setminus I$ . From Lemma 5.3 there exists  $i \in I$  such that  $\tilde{j} \in \mathcal{B}(i)$ . Then the following set is well defined and the following equality holds,

$$\tilde{I}(\tilde{j}) := \bigcap_{i \in I: \tilde{j} \in \mathcal{B}(i)} \mathcal{B}(i) = \{\tilde{k} \in \tilde{I} : \text{geod}(\tilde{j}, \tilde{k}) \cap (I \setminus \{\tilde{k}\}) = \emptyset\}. \quad (5.2)$$

The set  $\tilde{I}(\tilde{j})$  endowed with the set of edges  $\tilde{\mathcal{T}} \cap (\tilde{I}(\tilde{j}) \times \tilde{I}(\tilde{j}))$ , is the smallest subtree containing  $\tilde{j}$  and whose extremal points  $\mathcal{E}(\tilde{j}) = \{\tilde{k} \in \tilde{I}(\tilde{j}) : \tilde{k} \text{ has a unique neighbour in } \tilde{I}(\tilde{j})\}$  are all in  $I$ .

The property that every point in  $I$  has a finite number of  $U$ -neighbors supplies a good example for the next section. Observe that the sets  $\mathcal{V}(i)$  are finite for  $i \in I$ , is clearly equivalent to the fact that  $\mathcal{B}(i)$  are finite, for  $i \in I$ . This property can be easily expressed in terms of  $U$ .

**Lemma 5.4** *The sets  $\mathcal{B}(i)$  are finite for all  $i \in I$  if and only if*

$$\forall w \in \mathcal{W} \exists I^w \subset I \text{ finite such that: } \forall i \in I \setminus I^w, \max\{U_{ij} : j \in I^w, U_{ij} = U_{jj}\} > w. \quad (5.3)$$

**Proof.** Assume  $\mathcal{B}(i)$  are finite. Clearly, it is enough to prove (5.3) for large  $w \in \mathcal{W}$ . We shall assume that the finite set  $L = \{j \in I : U_{jj} \leq w\}$  is non empty and we define  $I^w = \cup_{j \in L} \mathcal{V}^*(j)$ .

Fix  $i_0 \in L$  as one of the closest points in  $I$  to the root  $\tilde{r}$ . For  $i \in I \setminus I^w$ , the geodesic  $\text{geod}(i, \tilde{r})$  must contain points on  $I^w$ , otherwise  $\text{geod}(i, i_0) = \{i, i_0\}$  which implies  $i \in \mathcal{V}^*(i_0)$ , a contradiction. Take  $k \in \text{geod}(i, \tilde{r}) \cap I^w$  the farthest point from  $\tilde{r}$ . It is clear that  $U_{ik} = U_{kk}$ . Assume  $U_{kk} \leq w$ , so  $k \in L$ . If  $\text{geod}(k, i) \cap I = \{k, i\}$  then  $i \in I^w$  which is a contradiction. Therefore, there is at least one  $m \in (\text{geod}(k, i) \cap I) \setminus \{i, k\}$ . Take  $m$  the closest of such points to  $k$ . Clearly  $m \in \mathcal{V}^*(k) \subseteq I^w$  contradicting the maximality of  $k$ . Then  $U_{kk} > w$ , proving the desired property.

Conversely, take  $i \in I$  and consider  $w = U_{ii}$ . We shall prove that  $\mathcal{V}^*(i) \subseteq I^w$ . In fact, take  $j \in \mathcal{V}^*(i) \setminus I^w$ . By hypothesis there is  $k \in I^w$  such that  $U_{kk} = U_{jk} > w$ . Since  $U_{jj} \geq U_{jk} = U_{kk} > w = U_{ii}$  and  $j \in \mathcal{V}^*(i)$ , we conclude  $k \in \text{geod}(i, j)$  and  $k \neq i$ . Since  $k \neq j$ , because  $k \in I^w$ , we arrive to a contradiction with the definition of  $\mathcal{V}^*(i)$ , proving the result.  $\square$

### 5.3 Generator and harmonic functions of an Ultrametric Matrix

In this section we associate to an ultrametric matrix  $U$  a  $q$ -matrix through its extension  $\tilde{U}$ . Consider the  $q$ -matrix  $\tilde{Q}$  given by (2.2), which satisfies  $\tilde{Q}\tilde{U} = \tilde{U}\tilde{Q} = -\mathbb{I}_{\tilde{I}}$ . We can also assume that  $\tilde{Q}$  is defined in  $\tilde{I} \cup \partial_{\tilde{r}}$  as in (2.3). Further, we consider  $\tilde{X}$  the Markov process associated to  $\tilde{Q}$  with lifetime  $\tilde{\zeta}$ .

We assume that  $\tilde{X}$  is transient. We denote by  $\tilde{\mu}$  the probability measure defined on  $\partial_{\infty}$ , the boundary of  $(\tilde{I}, \tilde{T})$ , that is proportional to the exit distribution of  $\tilde{X}$ .

Consider

$$\tau := \inf\{t > 0 : \tilde{X}_t \in I \cup \partial_{\tilde{r}}\} \wedge \tilde{\zeta},$$

We point out that  $\tilde{X}_{\tau}$  belongs to  $I \cup \partial_{\tilde{r}} \cup \partial_{\infty}$  with probability one. Notice that if  $\tilde{X}(0) = \tilde{j} \in \tilde{I} \setminus I$  then  $\tau = \inf\{t > 0 : \tilde{X}_t \in \mathcal{E}(\tilde{j}) \cup \partial_{\tilde{r}}\} \wedge \tilde{\zeta}$ .

Our main assumption is

$$\forall \tilde{j} \in \tilde{I} \setminus I : \mathbb{P}_{\tilde{j}}\{\tilde{X}_{\tau} \in I \cup \partial_{\tilde{r}}\} = 1. \quad (H4)$$

We can also write (H4) as  $\mathbb{P}_{\tilde{j}}\{\tau < \tilde{\zeta}\} = 1$  for every  $\tilde{j} \in \tilde{I} \setminus I$ . This is also equivalent to  $\mathbb{P}_{\tilde{j}}\{\tilde{X}_{\tau} \in \partial_{\infty}\} = 0$  for every  $\tilde{j} \in \tilde{I} \setminus I$ .

In the next Theorem we associate a  $q$ -matrix to a general ultrametric matrix verifying (H1)-(H4).

**Theorem 5.1** Assume  $U$  satisfies (H1)-(H4), then there exists a matrix  $Q : I \times I \rightarrow \mathbb{R}$  such that  $QU = UQ = -\mathbb{I}_I$ . Moreover  $Q_{ij} \neq 0$  if and only if  $j \in \mathcal{V}^*(i)$ , and we have

$$Q_{ij} = \tilde{Q}_{ij} + \sum_{\tilde{k} \in \tilde{I} \setminus I} \tilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\tilde{X}_\tau = j). \quad (5.4)$$

For  $i \neq j$  this formula takes the form

$$Q_{ij} = \tilde{Q}_{ij} \text{ if } (i, j) \in \tilde{\mathcal{T}} \text{ and } Q_{ij} = \tilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\tilde{X}_\tau = j) \text{ if } (i, j) \notin \tilde{\mathcal{T}}, j \in \mathcal{V}^*(i),$$

where  $\tilde{k} \in \tilde{I} \setminus I$  is the unique neighbor of  $i$  in  $\tilde{\mathcal{T}}$ , that belongs to  $\text{geod}(i, j)$ .

**Proof.** We set  $A = \tilde{Q}_{II}$ ,  $B = \tilde{Q}_{I, \tilde{I} \setminus I}$  and  $V = \tilde{U}_{\tilde{I} \setminus I, I}$ . Since  $\tilde{U}_{II} = U$  we get  $AU + BV = -\mathbb{I}_I$ .

The crucial step in the proof is to get a  $(\tilde{I} \setminus I) \times I$  matrix  $Z$  whose rows are summable and verifies  $ZU = V$ , which means

$$\tilde{U}_{\tilde{j}i} = \sum_{k \in I} Z_{\tilde{j}k} U_{ki}, \text{ for all } \tilde{j} \in \tilde{I} \setminus I, i \in I.$$

For any  $\tilde{j} \in \tilde{I} \setminus I$  consider the subtree  $\tilde{J} := \tilde{I}(\tilde{j})$  given by (5.2). We denote by  $\mathcal{E} \subset I$  the set of extremal points of  $\tilde{J}$ . Note that  $\tilde{J} \setminus \mathcal{E} \subseteq \tilde{I}$ . We consider the following  $q$ -matrix on  $\tilde{J} \times \tilde{J}$

$$C_{\tilde{l}\tilde{k}} = \tilde{Q}_{\tilde{l}\tilde{k}} \text{ if } \tilde{l} \in \tilde{J} \setminus \mathcal{E} \text{ and } C_{\tilde{l}\tilde{k}} = 0 \text{ otherwise.}$$

By definition of  $\tau$ , the Markov process induced by  $C$  is just the stopped process  $\tilde{X}^\tau$ . From the property  $\tilde{Q}\tilde{U} = -\mathbb{I}_{\tilde{J}}$  it is deduced that for each  $i \in I$  the restriction of  $U_{\bullet i}$  to  $\tilde{J}$ , is a  $C$ -harmonic function. Therefore,

$$U_{\tilde{j}i} = \mathbb{E}_{\tilde{j}}(U_{\tilde{X}_\tau i}) = \sum_{k \in \mathcal{E}} \mathbb{P}_{\tilde{j}}(\tilde{X}_\tau = k) U_{ki},$$

which gives the desired matrix  $Z$ . Since  $B$  is finitely supported and the rows of  $Z$  are summable we get

$$(A + BZ)U = -\mathbb{I}_I, \quad (5.5)$$

then  $Q = A + BZ$  should be the desired  $q$ -matrix. The explicit formula for  $Q$  is

$$Q_{ij} = \tilde{Q}_{ij} + \sum_{\tilde{k} \in \tilde{I} \setminus I} \tilde{Q}_{i\tilde{k}} Z_{\tilde{k}j} = \tilde{Q}_{ij} + \sum_{\tilde{k} \in \tilde{I} \setminus I} \tilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\tilde{X}_\tau = j). \quad (5.6)$$

From the structure of  $\tilde{Q}$  the last sum in (5.6) runs over  $\tilde{k} \in \tilde{I} \setminus I$  which are neighbors of  $i$  with respect to  $\tilde{\mathcal{T}}$ . From the shape of  $Z$  these values of  $\tilde{k}$  are further restricted to the set  $\mathcal{V}^*(j)$ . According to the Lemma 5.3 part (v) the set of such points is not empty when

$(i, j) \notin \tilde{\mathcal{T}}$  and moreover this set contains exactly one point  $\tilde{k} \in \tilde{I}$ . In summary, we have for  $i \neq j$

$$Q_{ij} = \tilde{Q}_{ij} \text{ if } (i, j) \in \tilde{\mathcal{T}} \text{ and } Q_{ij} = \tilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\tilde{X}_\tau = j) \text{ if } (i, j) \notin \tilde{\mathcal{T}}, j \in \mathcal{V}(i);$$

where in the last case,  $\tilde{k}$  is the unique neighbor of  $i$  in  $\tilde{\mathcal{T}}$  belonging to  $\text{geod}(i, j)$ . From this formula we deduce that for  $i \neq j$  we have  $Q_{ij} > 0$  if and only if  $j \in \mathcal{V}(i)$ . From (5.5) we deduce that  $Q_{ii} < 0$ . Also we get

$$Q_{ii} = \tilde{Q}_{ii} + \sum_{\tilde{k} \in \tilde{I}: (\tilde{k}, i) \in \tilde{\mathcal{T}}} \tilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\tilde{X}_\tau = i).$$

Now, let us prove that  $Q$  is a  $q$ -matrix. Let  $k \in \mathcal{V}(i)$  be such that  $U_{ki} = \min\{U_{ji} : j \in \mathcal{V}(i)\}$ . This minimum is attained because the set  $\{w \in \mathcal{W} : w \leq U_{ii}\}$  is finite. From the ultrametric property of  $U$  we have  $U_{jk} \geq \min\{U_{ji}, U_{ik}\} = U_{ik}$  for  $j \in \mathcal{V}^*(i)$ . Then, by using (5.5) we deduce that

$$0 \geq Q_{ii}U_{ik} + \sum_{j \in \mathcal{V}(i)} Q_{ij}U_{jk} \geq U_{ik}(\sum_{j \in I} Q_{ij}).$$

Hence  $Q$  is a  $q$ -matrix.

To finish the proof it is enough to show that  $Q$  is a symmetric matrix. This is equivalent to prove that

$$\tilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\tilde{X}_\tau = j) = \tilde{Q}_{j\tilde{l}} \mathbb{P}_{\tilde{l}}(\tilde{X}_\tau = i), \text{ for } j \in \mathcal{V}(i), (j, i) \notin \tilde{\mathcal{T}}, \quad (5.7)$$

where  $\tilde{k}$  (respectively  $\tilde{l}$ ) is the unique neighbor in  $\tilde{\mathcal{T}}$  of  $i$  (of  $j$  respectively) given by Lemma 5.3 part (v). The probabilities appearing in (5.7) can be computed in terms of  $\tilde{Y} = (\tilde{Y}_n)_{n \in \mathbb{N}}$ , the discrete skeleton of the Markov chain on  $\tilde{X}$  taking values on  $\tilde{I}$ . The transition probabilities for this chain are

$$\mathbb{P}(\tilde{Y}_1 = y_1 | \tilde{Y}_0 = y_0) = \frac{\tilde{Q}_{y_0 y_1}}{(-\tilde{Q}_{y_0 y_0})}.$$

If we define  $N = \min\{n \geq 0 : \tilde{Y}_n \in I \cup \{\partial_{\tilde{\tau}}\}\}$  then

$$\mathbb{P}_{\tilde{k}}(\tilde{X}_\tau = j) = \mathbb{P}_{\tilde{k}}(\tilde{Y}_N = j).$$

This last probability can be computed by summing up all possible trajectories  $\tilde{Y}_0 = \tilde{k}, \tilde{Y}_1 = y_1, \dots, \tilde{Y}_{n-2} = y_{n-2}, \tilde{Y}_{n-1} = \tilde{\ell}, \tilde{Y}_n = j$ , which do not visit  $I$  at any intermediate state. The probability of such trajectory is

$$\frac{\tilde{Q}_{\tilde{k} y_1}}{(-\tilde{Q}_{\tilde{k} \tilde{k}})} \frac{\tilde{Q}_{y_1 y_2}}{(-\tilde{Q}_{y_1 y_1})} \dots \frac{\tilde{Q}_{y_{n-2} \tilde{\ell}}}{(-\tilde{Q}_{y_{n-2} y_{n-2}})} \frac{\tilde{Q}_{\tilde{\ell} j}}{(-\tilde{Q}_{\tilde{\ell} \tilde{\ell}})}$$



The probability of the reverse trajectory  $\tilde{Y}_0 = \tilde{\ell}$ ,  $\tilde{Y}_1 = y_{n-2}, \dots, \tilde{Y}_{n-2} = y_1$ ,  $\tilde{Y}_{n-1} = \tilde{k}$ ,  $\tilde{Y}_n = i$ , is

$$\frac{\tilde{Q}_{\tilde{\ell}y_{n-2}}}{(-\tilde{Q}_{\tilde{\ell}\tilde{\ell}})} \frac{\tilde{Q}_{y_{n-2}y_{n-3}}}{(-\tilde{Q}_{y_{n-2}y_{n-2}})} \dots \frac{\tilde{Q}_{y_1\tilde{k}}}{(-\tilde{Q}_{y_1y_1})} \frac{\tilde{Q}_{\tilde{k}i}}{(-\tilde{Q}_{\tilde{k}\tilde{k}})}.$$

The symmetry of  $\tilde{Q}$  implies that (5.7) holds. Therefore,  $Q$  is symmetric and we deduce that  $UQ = -\mathbb{I}_I$ . This finishes the proof.  $\square$

As usual we say that a function  $h : I \rightarrow \mathbb{R}$  is  $Q$ -harmonic if  $Qh = 0$ . Our main result in relation with harmonic functions for ultrametric matrices is the following one.

**Theorem 5.2** *Assume  $U$  satisfies (H1)-(H4). Given a bounded  $Q$ -harmonic function  $h$  defined on  $I$  there exists a unique  $\tilde{Q}$ -harmonic function  $\tilde{h}$  defined on  $\tilde{I}$ , which is an extension of  $h$ .*

**Proof.** Consider the function

$$\tilde{h}(\tilde{i}) = \mathbb{E}_{\tilde{i}} \left( h(\tilde{X}_\tau) \right), \quad \tilde{i} \in \tilde{I}.$$

Clearly  $\tilde{h}$  is an extension of  $h$ . Using the strong Markov property for the time of first jump of  $\tilde{X}$  we deduce that  $\tilde{h}$  is  $\tilde{Q}$ -harmonic at every  $\tilde{j} \in \tilde{I} \setminus I$ . Now, for  $i \in I$  we have

$$\begin{aligned} \sum_{\tilde{j} \in \tilde{I}} \tilde{Q}_{ij} \tilde{h}(\tilde{j}) &= \sum_{j \in I} \tilde{Q}_{ij} h(j) + \sum_{\tilde{j} \in \tilde{I} \setminus I} \tilde{Q}_{ij} \tilde{h}(\tilde{j}) = \sum_{j \in I} \tilde{Q}_{ij} h(j) + \sum_{\tilde{j} \in \tilde{I} \setminus I} \tilde{Q}_{ij} \mathbb{E}_{\tilde{j}}(h(\tilde{X}_\tau)) \\ &= \sum_{j \in I} \tilde{Q}_{ij} h(j) + \sum_{\tilde{j} \in \tilde{I} \setminus I} \tilde{Q}_{ij} \left( \sum_{k \in I} \mathbb{P}_{\tilde{j}}(\tilde{X}_\tau = k) h(k) \right) \\ &= \sum_{j \in I} \left( \tilde{Q}_{ij} + \sum_{\tilde{k} \in \tilde{I} \setminus I} \tilde{Q}_{i\tilde{k}} \mathbb{P}_{\tilde{k}}(\tilde{X}_\tau = j) \right) h(j) = \sum_{j \in I} Q_{ij} h(j), \end{aligned}$$

where the last equality follows from (5.4). Since  $h$  is  $Q$ -harmonic we get  $\sum_{\tilde{j} \in \tilde{I}} \tilde{Q}_{ij} \tilde{h}(\tilde{j}) = 0$ .

Then  $\tilde{h}$  is  $\tilde{Q}$ -harmonic at  $i \in I$ .  $\square$

## 5.4 The Boundary of an Ultrametric Matrix

Recall that  $\tilde{\partial}_\infty$  can be identified with

$$\tilde{\partial}_\infty = \{(\tilde{i}_n : n \geq 0) : \tilde{i}_0 = \tilde{r}, \forall n \in \mathbb{N}, |\tilde{i}_n| = n \text{ and } (\tilde{i}_n, \tilde{i}_{n+1}) \in \tilde{\mathcal{T}}\}.$$

endowed with the topology generated by the sets  $\tilde{\mathcal{C}} = \{\tilde{\partial}_\infty(\tilde{i}) = [\tilde{i}, \infty] \cap \tilde{\partial}_\infty : \tilde{i} \in \tilde{I}\}$ . We denote by  $\tilde{\mathcal{F}}_\infty$  the associated  $\sigma$ -field. We will denote by  $\mathcal{F}_\infty$  the  $\sigma$ -field on  $\tilde{\partial}_\infty$  generated

by the sets  $\{\tilde{\partial}_\infty(i) : i \in I\}$ . We have  $\mathcal{F}_\infty \subseteq \tilde{\mathcal{F}}_\infty$ , and as we shall see further in an example, this inclusion can be strict.

The following definition of the boundary  $\partial_\infty^U$  associated to an ultrametric matrix extends the one for a tree. An infinite path  $(i_n : n \in \mathbb{N})$  in  $I$  is called a  $\preceq$ -chain if  $i_n \prec i_{n+1}$  for every  $n \in \mathbb{N}$ , and the  $\preceq$ -chain is maximal if we cannot add any element of  $I$  to it in order that it continues to be a  $\preceq$ -chain. In a tree a  $\preceq$ -chain  $(i_n : n \in \mathbb{N})$  is maximal, if and only if  $i_0 = r$  and  $|i_n| = n$  for every  $n \in \mathbb{N}$ . The boundary of  $I$  with respect to the ultrametric matrix  $U$  is defined as  $\partial_\infty^U = \{(i_n : n \in \mathbb{N}) \text{ is a maximal } \preceq\text{-chain}\}$ . We endowed  $\partial_\infty^U$  with the trace topology from  $\tilde{\partial}_\infty$ . From the equality

$$\partial_\infty^U = \bigcap_{n \geq 0} \left( \bigcup_{m \geq n} \bigcup_{i \in I : |i|=m} \{\xi \in \tilde{\partial}_\infty : \xi(m) = i\} \right),$$

we get that  $\partial_\infty^U \in \tilde{\mathcal{F}}_\infty$ .

The function given by

$$\mathbf{i}_\infty^U : \partial_\infty^U \rightarrow \tilde{\partial}_\infty, \mathbf{i}_\infty^U((i_n : n \geq 0)) = (\tilde{i}_n : n \geq 0) \Leftrightarrow \{i_n : n \geq 0\} \subseteq \{\tilde{i}_n : n \geq 0\}, \quad (5.8)$$

is a well-defined one-to-one function. We will identify  $\partial_\infty^U$  and  $\mathbf{i}_\infty^U(\partial_\infty^U)$ .

In general,  $\mathbf{i}_\infty^U$  is not onto as shows the following example.

**Examples.** Let  $A$  be a finite set (an alphabet), we set  $A^*$  the set of finite words. In particular the empty word, denoted by  $r$  is an element of  $A^*$ . The length of a word  $i$  is denoted by  $|i|$ , so  $|r| = 0$ . If  $|i| \geq 1$  and  $1 \leq m \leq |i|$  we denote by  $i[1, m]$  the set of first  $m$  coordinates of  $i$ . For any two words  $i, j$  we define the function  $N(i, j)$  by  $N(i, j) = 0$  if  $i = r$  or  $j = r$ , and  $N(i, j) = \max\{m \leq \min\{|i|, |j|\} : i[1, m] = j[1, m]\}$  when  $i$  and  $j$  are not  $r$ . Let  $w : \mathbb{N} \rightarrow \mathbb{R}_+$  be a positive strictly increasing function. For an infinite subset  $I \subseteq A^*$  we define the matrix  $U$  by  $U_{ij} = w(N(i, j))$ , for  $i, j \in I$ . In the sequel we fix  $A = \{0, 1, 2\}$ .

*Example 1.* Let  $I$  be the set of finite words finishing by 1. Then it is easy to see that the minimal tree extension can be identified with the rooted tree  $(\tilde{I}, \tilde{\mathcal{T}})$  where  $\tilde{I} = \{0, 1, 2\}^*$  and such that two points  $i, j$  are  $\tilde{\mathcal{T}}$ -neighbors if  $||i| - |j|| = 1$  and  $N(i, j) = \min\{|i|, |j|\}$ . Therefore  $\tilde{\partial}_\infty$  can be identified with  $\{0, 1, 2\}^\mathbb{N}$  and  $\partial_\infty^U$  with the set of infinite sequences in  $\{0, 1, 2\}^\mathbb{N}$  containing an infinite number of 1's. In this example  $\tilde{\mathcal{F}}_\infty$  does not coincide with  $\mathcal{F}_\infty$  on  $\tilde{\partial}_\infty$ , because in this last  $\sigma$ -field all the infinite sequences in  $\{0, 2\}^\mathbb{N}$  cannot be separated.

*Example 2.* Let  $I$  be the set of finite words of the form  $\{0, 2\}^*1$ , that is they finish by 1 and all the other letters are 0 or 2. Then in the minimal tree extension we can identify  $\tilde{I} = I$ , and  $\tilde{\partial}_\infty$  with  $\{0, 2\}^\mathbb{N}$ . Nevertheless,  $\partial_\infty^U$  is empty.  $\square$

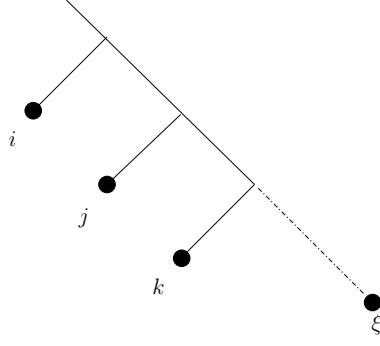


Figure 3:  $i, j, k \in I$ ;  $\xi \in \tilde{\partial}_\infty \setminus \partial_\infty^U$

Then, in general  $\partial_\infty^U$  is small compare to  $\tilde{\partial}_\infty$ , but as the following result shows, under (H4) it has full  $\tilde{\mu}$ -measure.

**Lemma 5.5** *Property (H4) is equivalent to  $\tilde{\mu}(\partial_\infty^U) = 1$ .*

**Proof.** First notice that if for some  $\tilde{j} \in \tilde{I} \setminus I$  it holds  $\mathbb{P}_{\tilde{j}}\{\tilde{X}_\tau \in I \cup \partial_{\tilde{\tau}}\} < 1$ , then  $\mathbb{P}_{\tilde{j}}\{\tilde{X}_{\tilde{\zeta}} \in \tilde{\partial}_\infty \setminus \partial_\infty^U\} > 0$ . Hence, the condition is necessary for (H4). For the reciprocal, assume that  $\tilde{\mu}(\tilde{\partial}_\infty \setminus \partial_\infty^U) > 0$ . Therefore there exists  $n \geq 0$  such that  $\mathbb{P}_{\tilde{\tau}}\{A_n\} > 0$ , where  $A_n = \cap_{m \geq n} \cap_{i \in I: |i|=m} \{\xi \in \tilde{\partial}_\infty : \xi(m) \neq i\}$ . Take any  $\tilde{i} \in \tilde{I} \setminus I$ ,  $|\tilde{i}| = n$  such that  $\tilde{\partial}_{\tilde{i}}(\infty) \cap A_n$  has positive  $\mathbb{P}_{\tilde{\tau}}$ -measure. Then we have  $\mathbb{P}_{\tilde{i}}\{\tilde{X}_{\tilde{\zeta}} \in \tilde{\partial}_\infty, \tilde{X}_{\tilde{\zeta}}(\ell) \notin I, \forall \ell \geq 0\} > 0$ , which contradicts hypothesis (H4).  $\square$

**Theorem 5.3** *Assume  $U$  satisfies (H1)-(H4). Let  $h$  be a bounded  $Q$ -harmonic function such that  $\lim_{i \rightarrow \xi} h(i) = \varphi(\xi)$  for every  $\xi \in \partial_\infty^U$ . Then, there exists  $\tilde{\varphi} = \lim_{\tilde{h}} \mu$ -a.e., where  $\tilde{h}$  is the harmonic function associated to  $h$  in Theorem 5.2. Moreover, if  $\tilde{\varphi}$  is in the domain of  $\tilde{W}^{-1}$ , then  $h$  has the representation*

$$h(i) = \int_{\tilde{\partial}_\infty} U_{i,\eta}(\tilde{W}^{-1}\tilde{\varphi})(\eta) \tilde{\mu}(d\eta). \quad (5.9)$$

**Proof.** From Lemma 5.5 we have  $\partial_\infty^U = \tilde{\partial}_\infty$ ,  $\mu$ -a.e. and therefore (almost) every point  $\xi \in \tilde{\partial}_\infty$  verifies  $|\{n \in \mathbb{N} : \xi(n) \in I\}| = \infty$ . Also, from the hypothesis there exists  $a = \lim_{\substack{n \rightarrow \infty \\ \xi(n) \in I}} h(\xi(n))$ . For the first part of the statement it suffices to show  $a = \lim_{n \rightarrow \infty} h(\xi(n))$ .

Let us consider the subsequence  $k(n) = \max\{m \leq n : \xi(m) \in I\}$ . We have  $\lim_{n \rightarrow \infty} k(n) = \infty$ .

On the other hand, for large  $n$ ,  $\mathcal{V}(\xi(n)) \subset [\xi(k(n)), \infty)$ , then  $\tilde{h}(\xi(n)) = \mathbb{E}_{\xi(n)}\left(h(\tilde{X}_\tau)\right)$  belongs to the convex closure of the set  $\{h(\xi(m)) : \xi(m) \in I, m \geq k(n)\}$ . Hence the result follows.

Now we are able to show relation (5.9). It suffices to notice that for every  $i \in I \cup \tilde{\partial}_\infty$  and  $\tilde{\mu}$ -a.e. in  $\eta \in \tilde{\partial}_\infty$ , it holds  $\tilde{U}_{i\eta} = U_{i\eta}$ . Then the proof follows from Corollary 3.1.  $\square$

**Remark 5.1** From a topological point of view  $\partial_\infty^U$  is dense in  $\tilde{\partial}_\infty$  if for all  $i \in I$  there exists  $j \in I, j \neq i$ , such that  $U_{ij} = U_{ii}$  (that is, if for all  $i \in I$  the set  $[i, \infty)^U$  is infinite). In fact, it suffices to note that by definition of the minimal tree, for every  $\xi \in \tilde{\partial}_\infty$  and  $n \geq 1$ , there exists some  $i \in I$  such that  $i \in \tilde{\partial}_\infty(\xi(n))$ . The desired density follows by taking any  $\eta \in \partial_\infty^U$  hanging from  $i$ .

## 6 The Process in the Boundary

In this section we describe the process at the boundary. In Theorem 6.1 we explicit the kernel of the process and in Theorem 6.2 we relate the behavior of the processes at different levels, that is when we killed it deeper and deeper in the tree. This allows to get exit times from the elementary pieces of the boundary, and further to construct a simulation of the process. We emphasize that no regularity on the tree is imposed.

### 6.1 Definition and Description of the Process

In the sequel we put  $Z \sim \exp[\lambda]$  to mean that  $Z$  is a random variable exponentially distributed with mean  $1/\lambda \in [0, \infty]$  and we denote  $B \sim \text{Ber}(a)$  a Bernoulli random variable  $B$  with  $\mathbb{P}\{B = 1\} = a \in [0, 1]$ .

First, let us describe the transition probability of the process at the boundary.

**Theorem 6.1** Consider the symmetric kernel

$$p(t, \xi, \eta) = \sum_{n=0}^{|\xi \wedge \eta|} \frac{e^{-t/G_n(\xi)} - e^{-t/G_{n+1}(\xi)}}{\mu(C^n(\xi))}, \quad (\xi, \eta) \in \partial_\infty^{\text{reg}} \times \partial_\infty^{\text{reg}}, \quad t > 0. \quad (6.1)$$

This kernel is sub-Markovian with total mass

$$e^{-t/G_0} = \int p(t, \xi, \eta) \mu(d\eta), \quad (6.2)$$

and it is also a Feller transition kernel.

The sub-Markov semigroup  $P_t^W f(\xi) = \int p(t, \xi, \eta) f(\eta) \mu(d\eta)$ , induced by this kernel in  $L^2(\mu)$  verifies

$$P_t^W f = \sum_{n \geq 0} e^{-t/G_n} \left( \mathbb{E}_\mu(f | \mathcal{F}_n) - \mathbb{E}_\mu(f | \mathcal{F}_{n-1}) \right).$$

The infinitesimal generator of this semigroup is an extension of  $-W^{-1}$  defined on  $\mathcal{D}$ , and its potential is  $W$ . Moreover the Green's kernel of this semigroup is  $U$ , that is

$$U_{\xi\eta} = \int_0^\infty p(t, \xi, \eta) dt \quad \text{for } \xi, \eta \in \partial_\infty.$$

**Proof.** We first notice that by integrating (6.1) we obtain  $e^{-t/G_0} = P_t^W \mathbf{1}$ , that is (6.2) holds. Consider the following family of operators acting on  $\mathcal{D}$

$$e^{-tW^{-1}} f := \lim_{k \rightarrow \infty} \left( I_{\mathcal{D}} - \frac{tW^{-1}}{k} \right)^k f = \sum_{n \geq 0} e^{-t/G_n} \left( \mathbb{E}_{\mu}(f|\mathcal{F}_n) - \mathbb{E}_{\mu}(f|\mathcal{F}_{n-1}) \right), \quad (6.3)$$

where the last equality follows from the fact that  $(G_n)$  is predictable, that is  $G_n$  is  $\mathcal{F}_{n-1}$ -measurable. Moreover since  $(G_n)$  is decreasing and positive we also obtain

$$\|e^{-tW^{-1}} f\|_2 \leq e^{-t/G_0} \|f\|_2.$$

Therefore,  $e^{-tW^{-1}}$  has a unique continuous extension to  $L^2(\mu)$  whose norm is bounded by  $e^{-t/G_0}$ . Clearly  $e^{-tW^{-1}} \mathbf{1} = e^{-t/G_0}$ , which implies that the norm of  $e^{-tW^{-1}}$  is  $e^{-t/G_0}$ . It can be also proven that  $(e^{-tW^{-1}} : t \geq 0)$  is a sub-Markovian semigroup acting on  $L^2(\mu)$ .

A simple computation yields for  $\xi \neq \eta$  and  $m \geq |\xi \wedge \eta|$

$$e^{-tW^{-1}} \mathbf{1}_{C^m(\eta)}(\xi) = \mu(C^m(\eta)) \sum_{n=0}^{|\xi \wedge \eta|} \frac{e^{-t/G_n(\xi)} - e^{-t/G_{n+1}(\xi)}}{\mu(C^n(\xi))} = \int p(t, \xi, \eta) \mathbf{1}_{C^m(\eta)}(\eta) \mu(d\eta). \quad (6.4)$$

In the case  $\mu(\{\xi^*\}) > 0$ , the series

$$\sum_{n=0}^{\infty} \frac{e^{-t/G_n(\xi^*)} - e^{-t/G_{n+1}(\xi^*)}}{\mu(C^n(\xi^*))} \leq \frac{1}{\mu(\{\xi^*\})} \sum_{n=0}^{\infty} e^{-t/G_n(\xi^*)} - e^{-t/G_{n+1}(\xi^*)} = \frac{e^{-t/G_0(\xi^*)}}{\mu(\{\xi^*\})},$$

is convergent and

$$e^{-tW^{-1}} \mathbf{1}_{\{\xi^*\}}(\xi) = \int p(t, \xi, \eta) \mathbf{1}_{\{\xi^*\}}(\eta) \mu(d\eta). \quad (6.5)$$

From equations (6.4) and (6.5) we deduce that

$$e^{-tW^{-1}} f = P_t^W f \quad \mu - a.e., \text{ for all } f \in \mathcal{D}.$$

Thus  $P_t^W$  is a pointwise representation of  $e^{-tW^{-1}}$  in  $L^2(\mu)$ .

Notice that from (3.14) the equalities

$$\int_0^{\infty} p(t, \xi, \eta) dt = \sum_{n=0}^{|\xi \wedge \eta|} (G_n(\xi) - G_{n+1}(\xi)) / \mu(C^n(\xi)) = U_{\xi\eta}$$

hold for all  $\xi, \eta$ . Therefore, for any  $f \geq 0$  in  $L^2(\mu)$  we have by Fubini's Theorem

$$\int_0^{\infty} P_t^W f(\xi) dt = \int \int_0^{\infty} p(t, \xi, \eta) dt f(\eta) \mu(d\eta) = \int U_{\xi\eta} f(\eta) \mu(d\eta) = W f(\xi).$$

Also a direct computation shows that for any  $f \in \mathcal{D}$

$$\frac{d}{dt} P_t^W f|_{t=0} = -W^{-1} f.$$

The Feller property of the transition kernel  $p$  is direct to check and it follows from the fact that for a simple function  $f$  we have  $P_t^W f$  is also simple (in particular continuous) and  $P_t^W f \rightarrow f$  as  $t \rightarrow 0$ .  $\square$

**Remark 6.1** *It is easy to show that for any  $t > 0$  fixed, the kernel  $p(t, \xi, \eta)$  given by (6.1) verifies the ultrametric inequality*

$$p(t, \xi, \eta) \geq \min\{p(t, \xi, \delta), p(t, \delta, \eta)\}, \text{ for every } \xi, \eta, \delta \in \partial_\infty^{reg}.$$

To the semigroup  $P^W$  we associate a Markov process denoted by  $(\Xi_t : 0 \leq t \leq \Upsilon)$ , where  $\Upsilon = \inf\{t > 0 : \Xi_t \notin \partial_\infty^{reg}\}$  is its lifetime. The coffin state of this process is written  $\dagger$ , that is  $\Xi_\Upsilon = \dagger$ . By  $\Xi^\nu$  we denote a copy of this Markov process with initial distribution  $\nu$  in  $\partial_\infty$  and when it starts from  $\xi$  we put  $\Xi^\xi := \Xi^{\delta_\xi}$ . The Feller property of  $p$  implies that  $\Xi$  has a right continuous with left limits version (see [9], Theorem I.9.4). We shall always take that version. On the other hand, and as we have already pointed out, by using Proposition 3.4 or by the arguments developed in [2] Theorem 4.1, the diffusive part in the Beurling-Deny formula vanishes, so  $\Xi$  is a pure jump process.

Let us describe more precisely the process  $\Xi$ , in which a main role is played by the killing times. Since the total mass verifies  $P_t^W \mathbf{1} = e^{-t/G_0}$ , the random time  $\Upsilon$  is exponentially distributed:  $\Upsilon \sim \exp[1/G_0]$ . By using this fact and the symmetry of the kernel  $p(t, \cdot, \cdot)$ , we can check that  $\mu$  is a quasi-stationary distribution for  $\Xi$ , that is

$$\mathbb{P}_\mu\{\Xi_t \in A\} = e^{-t/G_0} \mu(A) \text{ for any measurable } A \subseteq \partial_\infty. \quad (6.6)$$

We will interpret the formula (6.1) for the transition kernel  $p(t, \xi, \eta)$  in a recursive way. Let  $r_1 \in S_r$  be a successor of the root  $r$  such that  $\mu(\partial_\infty(r_1)) > 0$ . Let  $(\bar{I}, \bar{\mathcal{T}})$  be the subtree rooted by  $r_1$ , where  $\bar{I} = [r_1, \infty)$  and  $\bar{\mathcal{T}} = \mathcal{T} \cap \bar{I} \times \bar{I}$ . We add an absorbing state identified with  $r$  and we denote by  $\bar{\partial}_\infty = \partial_\infty(r_1)$  the boundary of the tree  $(\bar{I}, \bar{\mathcal{T}})$ . The induced level function is  $\|k \wedge l\| := |k \wedge l| - 1$ ,  $k, l \in \bar{I}$ . We also note that  $\bar{C}^n(\xi) = C^{n+1}(\xi)$  for  $\xi \in \bar{\partial}_\infty$ . Consider the tree matrix  $\bar{U}$  induced by the weight function  $\bar{\omega}_k$  satisfying the recursion

$$\bar{\omega}_{-1} = 0 \text{ and } \Delta_k(\bar{\omega}) = \mu(\bar{\partial}_\infty) \Delta_{k+1}(\omega) \text{ for } k \geq 0.$$

The limit probability measure on  $\bar{\partial}_\infty$  is given by the conditional measure  $\bar{\mu} = \mu(\bullet | \bar{\partial}_\infty)$ . From this definition the new sequence of  $\sigma$ -fields is  $\bar{\mathcal{F}}_n = \sigma(C^{n+1} \cap \bar{\partial}_\infty : C^{n+1} \in \mathcal{F}_n)$ . Also by definition  $\bar{G}_n = G_{n+1}$  for all  $n \geq 0$ , and  $(\bar{G}_n)$  is  $(\bar{\mathcal{F}}_n)$ -predictable.

From the strong Markov property we obtain for any measurable  $C \subseteq \bar{\partial}_\infty$

$$\mathbb{P}_{r_1}\{X_\zeta \in C, T_r = \infty\} = \mathbb{P}_{r_1}\{X_\zeta \in C\} - \mathbb{P}_{r_1}\{T_r < \infty\} \mathbb{P}_r\{X_\zeta \in C\}.$$

Also we have

$$\mathbb{P}_{r_1}\{X_\zeta \in C\} = \frac{\mathbb{P}_r\{X_\zeta \in C\}}{\mathbb{P}_r\{T_{r_1} < \infty\}}.$$

Hence, we deduce that

$$\mathbb{P}_{r_1}\{X_\zeta \in C, T_r = \infty\} = a\bar{\mu}(C),$$

where

$$a = \mathbb{P}_r\{X_\zeta \in \partial_\infty(r_1)\} \left( \frac{1}{\mathbb{P}_r\{T_{r_1} < \infty\}} - \mathbb{P}_{r_1}\{T_r < \infty\} \right).$$

The constant  $a$  is positive because  $\mu(\partial_\infty(r_1)) > 0$ . Then, the set of regular points  $\bar{\partial}_\infty^{reg}$  is exactly the set  $\partial_\infty^{reg} \cap \partial_\infty(r_1)$ .

Consider the operator  $\bar{W}$  acting on  $L^2(\bar{\partial}_\infty, \bar{\mu})$  given by

$$\bar{W}f(\xi) = \int \bar{U}_{\xi\eta}f(\eta)\bar{\mu}(d\eta)$$

Denote by  $(\bar{P}_t : t \geq 0)$  the semigroup associated to  $\bar{W}$ , and  $(\bar{\Xi}_t : 0 \leq t \leq \bar{\Upsilon})$  the induced Markov process on  $\bar{\partial}_\infty$  with coffin state  $\dagger$ . We denote by  $\bar{\Xi}^\xi$  a copy of  $\bar{\Xi}$  starting from  $\xi \in \bar{\partial}_\infty^{reg}$ .

The transition kernel for this semigroup in  $\bar{\partial}_\infty^{reg}$  is given by (see Theorem 6.1)

$$\bar{p}(t, \xi, \eta) = \sum_{n=0}^{\|\xi \wedge \eta\|} \frac{e^{-t/\bar{G}_n} - e^{-t/\bar{G}_{n+1}}}{\bar{\mu}(\bar{C}^n(\xi))} = \mu(\bar{\partial}_\infty)(p(t, \xi, \eta) - (e^{-t/G_0} - e^{-t/G_1})). \quad (6.7)$$

The total mass for this kernel is  $e^{-t/\bar{G}_0} = e^{-t/G_1}$  and therefore  $\bar{\Upsilon} \sim \exp[1/G_1]$ , that is  $\mathbb{P}_\xi\{\bar{\Upsilon} > t\} = e^{-t/G_1}$ .

**Theorem 6.2** *Fix  $\xi \in \bar{\partial}_\infty^{reg}$  and consider  $\bar{\Xi}^\xi, \Xi^\mu$  two random independent elements ( $\Xi^\mu$  is a copy of the process  $\Xi$  with initial distribution  $\mu$ ). Let  $B \sim \text{Ber}(1 - G_1/G_0)$  be a Bernoulli variable independent of  $\Xi^\mu$  and  $\bar{\Xi}^\xi$ . Under  $\mathbb{P}_\xi$  the following Markov process*

$$\tilde{\Xi}_t^\xi = \begin{cases} \bar{\Xi}_t^\xi & \text{if } t < \bar{\Upsilon} \\ \dagger & \text{if } t \geq \bar{\Upsilon} \text{ and } B = 0 \\ \Xi_{t-\bar{\Upsilon}}^\mu & \text{otherwise} \end{cases} \quad (6.8)$$

*is a copy of  $\Xi^\xi$  (that is  $\tilde{\Xi}^\xi$  and  $\Xi^\xi$  are identically distributed).*

**Proof.** For  $k \geq 1$  let  $\xi_1, \dots, \xi_k \in \partial_\infty^{reg}$ ,  $\xi_0 = \xi \in \bar{\partial}_\infty^{reg}$  and  $t_0 = 0 < t_1 < \dots < t_k < t_{k+1} = \infty$ . We must prove

$$\mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k\} = \mathbb{P}_\xi\{\Xi_{t_u} \in d\xi_u, u = 1, \dots, k\}. \quad (6.9)$$

Let us study the case  $k = 1$ . For  $\eta \in \partial_\infty^{reg} \setminus \bar{\partial}_\infty$  we have

$$\mathbb{P}_\xi\{\tilde{\Xi}_t \in d\eta\} = \left(1 - \frac{G_1}{G_0}\right) \int_0^t \mathbb{P}_\mu\{\Xi_{t-u} \in d\eta\} \frac{e^{-u/G_1}}{G_1} du = \mu(d\eta) \left(\frac{1}{G_1} - \frac{1}{G_0}\right) \int_0^t e^{-(t-u)/G_0} e^{-u/G_1} du,$$

where the last equality follows from the fact that  $\mu$  is quasi-stationary for  $\Xi$  (see (6.6)). From (6.1) we obtain

$$\mathbb{P}_\xi\{\tilde{\Xi}_t \in d\eta\} = \mu(d\eta)(e^{-t/G_0} - e^{-t/G_1}) = \mathbb{P}_\xi\{\Xi_t \in d\eta\}.$$

Now, when  $\eta \in \bar{\partial}_\infty^{reg}$ , we get again from (6.6)

$$\begin{aligned} \mathbb{P}_\xi\{\tilde{\Xi}_t \in d\eta\} &= \mathbb{P}_\xi\{\bar{\Xi}_t \in d\eta\} + (1 - \frac{G_1}{G_0}) \int_0^t \mathbb{P}_\mu\{\Xi_{t-u} \in d\eta\} \frac{e^{-u/G_1}}{G_1} du \\ &= \bar{p}(t, \xi, \eta) \bar{\mu}(d\eta) + (e^{-t/G_0} - e^{-t/G_1})\mu(d\eta). \end{aligned}$$

Thus, from (6.7) we find

$$\bar{p}(t, \xi, \eta) \bar{\mu}(d\eta) = (p(t, \xi, \eta) - (e^{-t/G_0} - e^{-t/G_1}))\mu(d\eta) \text{ for } \xi, \eta \in \bar{\partial}_\infty^{reg}, \quad (6.10)$$

so

$$\mathbb{P}_\xi\{\tilde{\Xi}_t \in d\eta\} = p(t, \xi, \eta) \mu(d\eta) = \mathbb{P}_\xi\{\Xi_t \in d\eta\},$$

showing the case  $k = 1$ .

Assume that  $k \geq 2$ . By a recursive argument it is sufficient to show

$$\mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k\} = \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k-1\} p(t_k - t_{k-1}, \xi_{k-1}, \xi_k). \quad (6.11)$$

It is useful to consider the set  $\mathcal{K} = \{u \leq k : \xi_u \in \partial_\infty^{reg} \setminus \bar{\partial}_\infty\}$ . If  $\mathcal{K} = \emptyset$  we define  $\ell = k + 1$  so  $t_\ell = \infty$ . Otherwise we put  $\ell = \min \mathcal{K}$ . We have

$$\mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k\} = \int_0^{t_\ell} \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k; \bar{\Upsilon} \in dt\}.$$

Observe that from the definition of  $(\tilde{\Xi}_t)$  we also have

$$\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k; \bar{\Upsilon} > t_k\} = \{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k\}. \quad (6.12)$$

(I). Let us assume  $\ell \leq k$ . By definition of  $(\tilde{\Xi}_t)$  we find

$$\begin{aligned} &\mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k\} \\ &= \sum_{j=1}^{\ell} \int_{t_{j-1}}^{t_j} \mathbb{P}_\mu\{\Xi_{t_u-t} \in d\xi_u, u = j, \dots, k\} e^{-t/G_1} (G_1^{-1} - G_0^{-1}) \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, j-1\} dt, \end{aligned}$$

where it is implicit that  $\mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, j-1\} = 1$  if  $j = 1$ . From (6.6), we get

$$\forall t \in (t_{j-1}, t_j) : \mathbb{P}_\mu\{\Xi_{t_u-t} \in d\xi_u, u = j, \dots, k\} = \mu(d\xi_j) e^{-(t_j-t)/G_0} \mathbb{P}_{\xi_j}\{\Xi_{t_u-t_j} \in d\xi_u, u = j+1, \dots, k\}.$$



If  $j \leq k-1 \wedge \ell$  and  $t \in (t_{j-1}, t_j)$ , we can use the Markov property of  $\Xi_t$  to obtain,

$$\begin{aligned} & \mathbb{P}_\mu\{\Xi_{t_u-t} \in d\xi_u, u=j, \dots, k\} \\ &= \mu(d\xi_j) e^{-(t_j-t)/G_0} \mathbb{P}_{\xi_j}\{\Xi_{t_u-t_j} \in d\xi_u, u=j+1, \dots, k-1\} p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) \mu(d\xi_k) \\ &= \mathbb{P}_\mu\{\Xi_{t_u-t} \in d\xi_u, u=j, \dots, k-1\} p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) \mu(d\xi_k). \end{aligned}$$

Then

$$\begin{aligned} & \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k; \bar{\Upsilon} \leq t_{k-1 \wedge \ell}\} = \\ & \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1; \bar{\Upsilon} \leq t_{k-1 \wedge \ell}\} p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) \mu(d\xi_k). \end{aligned} \quad (6.13)$$

In the case  $\ell \leq k-1$  these last estimates lead to

$$\mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k\} = \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1\} p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) \mu(d\xi_k),$$

so relation (6.11) is verified.

To finish case **(I)** we assume  $\ell = k$ . By decomposing on the events  $\{\bar{\Upsilon} < t_{k-1}\}$  and  $\{\bar{\Upsilon} \in (t_{k-1}, t_k)\}$  we find

$$\begin{aligned} & \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k\} = \\ & \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1, \bar{\Upsilon} \leq t_{k-1}\} p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) \mu(d\xi_k) + \\ & \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1\} \left( \int_{t_{k-1}}^{t_k} e^{-(t-t_{k-1})/G_1} (G_1^{-1} - G_0^{-1}) e^{-(t_k-t)/G_0} dt \right) \mu(d\xi_k). \end{aligned}$$

Now, we use (6.12) to obtain

$$\begin{aligned} & \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1, \bar{\Upsilon} \leq t_{k-1}\} = \\ & \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1\} - \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1\}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k\} &= \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1\} p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) \mu(d\xi_k) \\ &\quad - A(t_{k-1}, t_k, \xi_{k-1}, \xi_k) \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1\} \mu(d\xi_k), \end{aligned}$$

with

$$A(t_{k-1}, t_k, \xi_{k-1}, \xi_k) = p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) + e^{-\frac{t_k}{G_0} + \frac{t_{k-1}}{G_1}} \left( e^{-t_{k-1}(\frac{1}{G_1} - \frac{1}{G_0})} - e^{-t_k(\frac{1}{G_1} - \frac{1}{G_0})} \right).$$

From (6.1) and since  $|\xi_{k-1} \wedge \xi_k| = 0$  we have  $p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) = e^{-(t_k - t_{k-1})/G_0} - e^{-(t_k - t_{k-1})/G_1}$ . A simple computation gives  $A(t_{k-1}, t_k, \xi_{k-1}, \xi_k) = 0$ . Hence, we have shown

$$\mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k\} = \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u=1, \dots, k-1\} p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) \mu(d\xi_k),$$

and equality (6.11) holds.

(II) Assume  $\ell = k + 1$ . By decomposing on the events  $\{\bar{\Upsilon} < t_{k-1}\}$ ,  $\{\bar{\Upsilon} \in (t_{k-1}, t_k)\}$  and  $\{\bar{\Upsilon} > t_k\}$  we obtain

$$\begin{aligned} \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k\} = \\ \mathbb{P}_\xi\{\tilde{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k-1\}p(t_k - t_{k-1}, \xi_{k-1}, \xi_k)\mu(d\xi_k) - \\ \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k-1\}p(t_k - t_{k-1}, \xi_{k-1}, \xi_k)\mu(d\xi_k) + \\ \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k-1\}e^{-\frac{t_k}{G_0} + \frac{t_{k-1}}{G_1}} \left( e^{-t_{k-1}(\frac{1}{G_1} - \frac{1}{G_0})} - e^{-t_k(\frac{1}{G_1} - \frac{1}{G_0})} \right) \mu(d\xi_k) + \\ \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k\}. \end{aligned}$$

From equality (6.10) we have

$$\begin{aligned} \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k\} &= \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k-1\}\bar{p}(t_k - t_{k-1}, \xi_{k-1}, \xi_k)\bar{\mu}(d\xi_k) \\ &= \mathbb{P}_\xi\{\bar{\Xi}_{t_u} \in d\xi_u, u = 1, \dots, k-1\} \left( p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) - \left( e^{-\frac{t_k - t_{k-1}}{G_0}} - e^{-\frac{t_k - t_{k-1}}{G_1}} \right) \right) \mu(d\xi_k). \end{aligned}$$

Hence, the proof is finished because  $A'(t_{k-1}, t_k, \xi_{k-1}, \xi_k) = 0$  with

$$\begin{aligned} A'(t_{k-1}, t_k, \xi_{k-1}, \xi_k) &= -p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) + e^{-\frac{t_k}{G_0} + \frac{t_{k-1}}{G_1}} \left( e^{-t_{k-1}(\frac{1}{G_1} - \frac{1}{G_0})} - e^{-t_k(\frac{1}{G_1} - \frac{1}{G_0})} \right) \\ &\quad + p(t_k - t_{k-1}, \xi_{k-1}, \xi_k) - \left( e^{-\frac{t_k - t_{k-1}}{G_0}} - e^{-\frac{t_k - t_{k-1}}{G_1}} \right). \end{aligned}$$

□

Let us define the iterated of the above procedure. We fix  $\xi^* \in \partial_\infty^{reg}$  a regular point of the boundary and consider the points  $\xi^*(n)$  in its geodesic starting at the root. Let  ${}^n\mu = \mu(\bullet \mid C^n(\xi^*))$  be the conditional measure to  $C^n(\xi^*)$  and  ${}^nU$  be the tree matrix induced by the weight function  ${}^n\omega$  satisfying the recursion

$${}^n\omega_{-1} = 0 \text{ and } \Delta_k({}^n\omega) = \mu(C^n(\xi^*))\Delta_{k+n}(\omega) \text{ for } k \geq 0.$$

Consider the operator

$${}^nWf(\xi) = \int {}^nU_{\xi\eta} f(\eta) {}^n\mu(d\eta) \text{ on } L^2(\partial_\infty(\xi^*(n)), {}^n\mu).$$

Denote by  ${}^n\Xi$  the process with generator  $-({}^nW)^{-1}$  and coffin state  ${}^n\ddagger$ . When the process starts from the distribution  $\nu$  we put  ${}^n\Xi^\nu$ . So  ${}^n\Xi^\xi$  denotes a version of the process starting at  $\xi \in \partial_\infty^{reg}(\xi^*(n))$ . With this notation  ${}^0\Xi = \Xi$  and  ${}^1\Xi = \bar{\Xi}$ .

The lifetime of  ${}^n\Xi$  is written  ${}^n\Upsilon$  which verifies  ${}^n\Upsilon \sim \exp[1/G_n]$ . We have  ${}^0\Upsilon = \Upsilon$ ,  ${}^1\Upsilon = \bar{\Upsilon}$ . For  $\xi \in \partial_\infty^{reg}(\xi^*(n+1))$  it holds  $\mathbb{P}_\xi\{{}^n\Upsilon \geq {}^{n+1}\Upsilon\} = 1$  and

$$\mathbb{P}_\xi\{{}^n\Upsilon > t > {}^{n+1}\Upsilon\} = e^{-t/G_n} - e^{-t/G_{n+1}}.$$

The variable  $\Upsilon$  is the exit time of  $\Xi$  from  $\partial_\infty^{reg}$ , but for  $n \geq 1$ ,  ${}^n\Upsilon$  is not the exit time of  $\Xi$  from  $C^n(\xi)$ . We write  $\mathcal{R}_n := \inf\{t > 0 : \Xi_t \notin C^n(\xi^*)\}$ , the exit time from  $C^n(\xi^*)$ .

**Proposition 6.1** *The exit time  $\mathcal{R}_n$  from  $C^n(\xi^*)$  starting from a regular point  $\xi \in C^n(\xi^*)$  is exponentially distributed with parameter*

$$\beta_n(\xi^*) = \mu(C^n(\xi^*)) \left[ \frac{1}{G_0(\xi^*)} + \sum_{k=1}^n \frac{1}{G_k(\xi^*)} \left( \frac{1}{\mu(C^k(\xi^*))} - \frac{1}{\mu(C^{k-1}(\xi^*))} \right) \right] \quad (6.14)$$

that is  $\mathbb{P}_\xi\{\mathcal{R}_n > t\} = e^{-t\beta_n(\xi^*)}$ .

**Proof.** For  $n = 0$ ,  $\mathcal{R}_0$  is the lifetime of  $\Xi$  which is exponentially distributed with parameter  $\beta_0 = 1/G_0$ . Now, we will do the computation only the case  $n = 1$ , the general case is proven analogously. From (6.8) we compute the distribution of  $\mathcal{R}_1$  by

$$\mathbb{P}_\xi\{\mathcal{R}_1 > t\} = e^{-t/G_1} + \left(1 - \frac{G_1}{G_0}\right) \int_0^t \frac{e^{-u/G_1}}{G_1} \int_{\bar{\partial}_\infty} \mathbb{P}_\eta\{\mathcal{R}_1 > t - u\} \mu(d\eta) du. \quad (6.15)$$

Integrating this relation with respect to  $\xi \in \bar{\partial}_\infty$  we obtain the following equation for  $\psi(t) = \int_{\bar{\partial}_\infty} \mathbb{P}_\eta\{\mathcal{R}_1 > t\} \mu(d\eta)$

$$\psi(t) = \mu(\bar{\partial}_\infty) \left( e^{-t/G_1} + \left(1 - \frac{G_1}{G_0}\right) \int_0^t \frac{e^{-u/G_1}}{G_1} \psi(t - u) du \right).$$

The solution to this equation is given by  $\psi(t) = \mu(\bar{\partial}_\infty)e^{-t\beta_1}$ , where

$$\beta_1 = \frac{1 - (1 - \frac{G_1}{G_0})\mu(\bar{\partial}_\infty)}{G_1} = \frac{\mu(\partial_\infty \setminus \bar{\partial}_\infty)}{G_1} + \frac{\mu(\bar{\partial}_\infty)}{G_0} \in \left(\frac{1}{G_1}, \frac{1}{G_0}\right).$$

Replacing this expression on the right hand side of (6.15) we obtain  $\mathbb{P}_\xi\{\mathcal{R}_1 > t\} = e^{-t\beta_1}$ .  $\square$

**Remark 6.2** *We notice that  $\beta_n(\xi) = \beta_n(\xi^*)$  for all regular points  $\xi \in C^n(\xi^*)$ .*

In what follows we explain in detail a scheme for simulating the process  $\Xi$  using exponential random variables, and a natural generalization of Theorem 6.2. In this result we denote by  ${}_0\tilde{\Xi}$  a copy of  $\Xi$ .

For an approximation of the process  $\Xi$  using the projections of its generator onto the spaces associated to the filtration defined by the levels of the tree see [32].

**Theorem 6.3** *Let  $n \geq 1$ ,  $\xi \in \partial_\infty$  and  $(B_k : k \geq 1)$  be a sequence of independent Bernoulli random variables with  $\mathbb{P}_\xi\{B_k = 1\} = 1 - \mathbb{P}_\xi\{B_k = 0\} = 1 - G_k(\xi)/G_{k-1}(\xi)$ . Then, under  $\mathbb{P}_\xi$  the following Markov process, defined recursively,*

$${}_n\tilde{\Xi}_t^\xi = \begin{cases} {}^n\Xi_t^\xi & \text{if } t < {}^n\Upsilon \\ \dagger & \text{if } t \geq {}^n\Upsilon \text{ and } B_k = 0 \text{ for } 1 \leq k \leq n \\ {}^k\tilde{\Xi}_{t-{}^n\Upsilon}^{k\mu} & \text{if } t \geq {}^n\Upsilon \text{ and } B_{k+1} = 1, B_p = 0 \text{ for } k+1 < p \leq n, \end{cases} \quad (6.16)$$

is a copy of  $\Xi^\xi$  (recall that  ${}^k\mu = \mu(\bullet | C^k(\xi))$ ).

Therefore if we could define properly  $\lim_{n \rightarrow \infty} {}^n\tilde{\Xi}$ , we would get this limit is also distributed as  $\Xi$ . We will achieve this by using a backward construction of the process  $\Xi$ . First we state a result on exponential variables whose proof is left to the reader.

**Lemma 6.1** *Let  $0 < \lambda_0 < \lambda_1$ .*

(i) *Let  $\Theta_1$ ,  $\Theta_0$  and  $B$  be independent random variables such that  $\Theta_1 \sim \exp[\lambda_1]$ ,  $\Theta_0 \sim \exp[\lambda_0]$  and  $B \sim \text{Ber}(1 - \lambda_0/\lambda_1)$ . Then the variable  $\Gamma_0 = \Theta_1 + B\Theta_0$  is distributed as  $\exp[\lambda_0]$ .*

(ii) *Let  $\Gamma_0$ ,  $\Gamma'_0$  and  $Z_1$  be independent random variables such that  $\Gamma_0 \sim \Gamma'_0 \sim \exp[\lambda_0]$  and  $Z_1 \sim \exp[\lambda_1 - \lambda_0]$ . Consider the random vector  $(\Theta_1, \Theta_0, B)$  defined in the following conditional way*

$$\Theta_1 = \Gamma_0, \Theta_0 = \Gamma'_0, B = 0 \text{ if } Z_1 \geq \Gamma_0 \text{ and } \Theta_1 = Z_1, \Theta_0 = \Gamma_0 - Z_1, B = 1 \text{ if } Z_1 < \Gamma_0.$$

*Then  $\Theta_1$ ,  $\Theta_0$  and  $B$  are independent random variables that verify  $\Theta_1 \sim \exp[\lambda_1]$ ,  $\Theta_0 \sim \exp[\lambda_0]$ ,  $B \sim \text{Ber}(1 - \lambda_0/\lambda_1)$  and  $\Gamma_0 = \Theta_1 + B\Theta_0$ .*

Now we introduce the elements involved in the simulation of the process. First, for  $t > 0$  we will denote by  $\mathbf{K}_t {}^n\Xi^\xi$  a copy of the process  ${}^n\Xi^\xi$ , conditioned to the fact that the killing time  ${}^n\Upsilon$  verifies  ${}^n\Upsilon = t$ . In particular  $\mathbf{K}_t \Xi^\xi$  denotes a copy of the process  $\Xi^\xi$ , conditioned to be killed at time  $\Upsilon = t$ .

Now, consider the following set of sites

$$\mathcal{M} = \{\vec{k} = (k_0, \dots, k_n) : 0 = k_0 \leq \dots \leq k_n, k_i \in \mathbb{N}, n \in \mathbb{N}\}.$$

Let  $\vec{k} = (k_0, \dots, k_n) \in \mathcal{M}$ . We put  $|\vec{k}| = n$  and call it the length of  $\vec{k}$ .  $\mathcal{M}$  is the set of sites of a tree with root  $(0)$  and where every site  $\vec{k}$  has a countable number of successors  $(\vec{k}, m) = (k_0, \dots, k_n, m)$  with  $m \geq k_n$ . If  $|\vec{k}| \geq 1$  we denote by  $\vec{k}^- = (k_0, \dots, k_{n-1})$  its predecessor. We call level  $n$  the class of sites with length  $n$ . We define  $\vec{k} + 1$  as follows,

$$(0) + 1 = (1) \text{ and } \vec{k} + 1 = (\vec{k}^-, k_n + 1) \text{ for } |\vec{k}| = n \geq 1.$$

We denote  $\mathcal{M} + 1 = \{\vec{k} + 1 : \vec{k} \in \mathcal{M}\}$ . Observe that  $\vec{k} + 1$  is in  $\mathcal{M}$  except in the case  $\vec{k} = (0)$ . On the other hand, if  $\vec{k} \in \mathcal{M}$  and  $|\vec{k}| = n \geq 1$ , then  $\vec{k} \in \mathcal{M} + 1$  if and only if  $k_n > k_{n-1}$ . We put  $|(1)| = 0$  so  $|\vec{k} + 1| = |\vec{k}|$  holds for all  $\vec{k} \in \mathcal{M}$ .

Now, we will define a countable random set of points  $\Lambda = \left(\Lambda(\vec{k}) : \vec{k} \in \mathcal{M} + 1\right)$  taking values in  $\partial_\infty$ . We will do it in a recursive way on the length of  $\vec{k}$ . First we fix

$$\Lambda((1)) = \xi \in \partial_\infty,$$

For the other levels these random variables satisfy the following conditional laws. Let  $n \geq 0$ . For level  $n + 1$  and  $|\vec{k}| = n$  we put,

$$\mathbb{P}\{\Lambda((\vec{k}, m)) \in A_m : m > k_n \mid \Lambda(\vec{k}'), \vec{k}' \in \mathcal{M} + 1, |\vec{k}'| \leq n\} = \prod_{m > k_n} \mu\{A_m \mid C^{m-1}(\Lambda(\vec{k} + 1))\};$$

with  $A_m$  a measurable set,  $A_m \subseteq C^{m-1}(\Lambda(\vec{k} + 1))$ ,  $m \geq 1$ . In particular  $\Lambda((\vec{k}, m))$  given  $\Lambda(\vec{k}'), \vec{k}' \in \mathcal{M} + 1, |\vec{k}'| \leq n$ , is distributed according to  $\mu(\cdot | C^{m-1}(\Lambda(\vec{k} + 1)))$ .

Conditionally on  $\Lambda$ , we consider the following countable family of independent random variables  $(\mathbf{Z}_m^{\vec{k}} : \vec{k} = (k_0, \dots, k_n) \in \mathcal{M}, m > k_n)$ , whose marginal distributions verify

$$\mathbf{Z}_m^{\vec{k}} \sim \exp \left[ \frac{1}{G_m(\Lambda(\vec{k} + 1))} - \frac{1}{G_{m-1}(\Lambda(\vec{k} + 1))} \right], \quad m > k_n.$$

**Lemma 6.2** *We have*

$$\mathbb{P}\{\forall \vec{k} \in \mathcal{M} : \liminf_{m \rightarrow \infty} \mathbf{Z}_m^{\vec{k}} = 0 \mid \Lambda\} = 1 \quad (6.17)$$

**Proof.** Since  $\Lambda(\vec{k})$  is a regular point, we get  $G_m(\Lambda(\vec{k} + 1)) > G_{m+1}(\Lambda(\vec{k} + 1))$  and  $\lim_{n \rightarrow \infty} 1/G_n(\Lambda(\vec{k} + 1)) = \infty$ . Then, the telescopic property gives

$$\forall x > 0, \forall m_0 > k_n : \mathbb{P}\{\forall m \geq m_0 : \mathbf{Z}_m^{\vec{k}} \geq x \mid \Lambda\} = \prod_{m \geq m_0} e^{-x \left( \frac{1}{G_m(\Lambda(\vec{k} + 1))} - \frac{1}{G_{m-1}(\Lambda(\vec{k} + 1))} \right)} = 0.$$

Therefore

$$\mathbb{P}\{\liminf_{m \rightarrow \infty} \mathbf{Z}_m^{\vec{k}} = 0 \mid \Lambda\} = 1.$$

Since  $\mathcal{M}$  is a countable set, the result is shown.  $\square$

Hence, we can assume  $\liminf_{m \rightarrow \infty} \mathbf{Z}_m^{\vec{k}} = 0$ . We will denote  $\mathbf{Z}^{\vec{k}} = (\mathbf{Z}_m^{\vec{k}} : m > k_n)$ . In particular  $\mathbf{Z}^{(0)} = (\mathbf{Z}_m^{(0)} : m > 0)$ .

Now, we need to introduce some operations in the class of strictly positive sequences having 0 as an accumulation point. Let  $\ell \geq 0$  be a positive integer,  $b > a \geq 0$  and  $\mathbf{z} = (\mathbf{z}_n : n > \ell)$  be a strictly positive sequence verifying  $\liminf_{n \rightarrow \infty} \mathbf{z}_n = 0$ . Consider the strictly increasing sequence  $(\underline{\gamma}_m := \underline{\gamma}_m[\mathbf{z}, \ell; a, b] : m \geq \ell)$  given by

$$\begin{aligned} \underline{\gamma}_\ell &:= \ell, \quad \underline{\gamma}_{\ell+1} := \inf\{n > \underline{\gamma}_\ell : \mathbf{z}_n < b - a\} \text{ and} \\ \underline{\gamma}_{m+1} &:= \inf\{n > \underline{\gamma}_m : \mathbf{z}_n < \mathbf{z}_{\underline{\gamma}_m}\} \text{ for } m > \ell + 1. \end{aligned}$$

Associated to it, we define a new sequence  $\mathbf{z}' := \mathbf{z}[\ell; a, b]$  whose elements  $\mathbf{z}' = (\mathbf{z}'_n : n \geq \ell)$  are given by

$$\mathbf{z}'_\ell = b \text{ and } \mathbf{z}'_m = a + \mathbf{z}_{\underline{\gamma}_m} \text{ for } m > \ell. \quad (6.18)$$

We also introduce the following sequence of integers

$$\bar{\gamma}_m = \bar{\gamma}_m[\mathbf{z}, \ell; a, b] := \underline{\gamma}_{m+1} - 1 \text{ for } m \geq \ell.$$

Therefore  $\underline{\gamma}_m \leq \bar{\gamma}_m$ . Notice that the sequence  $(\mathbf{z}'_n : n \geq \ell)$  strictly decreases to  $a$ .

The next step consists in defining a countable random set of times  $\mathbf{t} = \{\mathbf{t}_{\vec{k}} : \vec{k} \in \mathcal{M} \cup \{(1)\}\}$  taking values in  $\mathbb{R}_+$ , conditioned to  $\Lambda$ . Also, to each point  $\mathbf{t}_{\vec{k}}$  we associate a point  $\xi_{\vec{k}} \in \partial_\infty$ .

This construction will be done in a recursive way on the levels of  $\mathcal{M}$ . For level 0 we put  $\mathbf{t}_{(1)} = 0$  and we choose  $\mathbf{t}_{(0)} \sim \exp[1/G_0]$ . We will also put  $\underline{\gamma}_0 = 0$ . We define  $\xi_{(1)} = \xi$  and  $\xi_{(0)} = \dagger$ .

Let us define  $\mathbf{t}_{\vec{k}}$  for level 1, that is when  $|\vec{k}| = 1$ . From Lemma 6.2 we have  $\liminf_{m \rightarrow \infty} \mathbf{Z}_m^{\vec{k}} = 0$ , then we can define

$$\mathbf{t}_{(0,m)} = \mathbf{Z}'_m \text{ where } \mathbf{Z}' = \mathbf{Z}^{(0)}[0; 0, \mathbf{t}_{(0)}].$$

Therefore the sequence  $\mathbf{t}_{(0,m)}$  starts from  $\mathbf{t}_{(0,0)} = \mathbf{t}_{(0)}$  and it is strictly decreasing to 0. We introduce the sequences

$$\underline{\gamma}_m^{(0)} = \underline{\gamma}_m[\mathbf{Z}^{(0)}, 0; 0, \mathbf{t}_{(0)}] \text{ and } \bar{\gamma}_m^{(0)} = \bar{\gamma}_m[\mathbf{Z}^{(0)}, 0; 0, \mathbf{t}_{(0)}] \text{ for } m \geq 0.$$

By definition  $\underline{\gamma}_m^{(0)} \leq \bar{\gamma}_m^{(0)} = \underline{\gamma}_{m+1}^{(0)} - 1$ ,  $\underline{\gamma}_0^{(0)} = 0$ ,  $\underline{\gamma}_1^{(0)} = \inf\{m > 0 : \mathbf{Z}_m^{(0)} < \mathbf{t}_{(0)}\}$  and

$$\mathbf{t}_{(0,m)} = \mathbf{Z}_{\underline{\gamma}_m^{(0)}}^{(0)}, \text{ for } m \geq 1.$$

We associate to  $\mathbf{t}_{(0,0)}$  the value  $\xi_{(0,0)} = \xi_{(0)} = \dagger$  and for  $m \geq 1$  we associate to each  $\mathbf{t}_{(0,m)}$  the value  $\xi_{(0,m)} = \Lambda((0, \underline{\gamma}_m^{(0)}))$ . In each interval  $[\mathbf{t}_{(0,m)}, \mathbf{t}_{(0,m-1)})$  we put a copy of the process

$$\mathbf{K}_{\mathbf{t}_{(0,m-1)} - \mathbf{t}_{(0,m)}} \bar{\gamma}_{m-1}^{(0)} \Xi^{\xi_{(0,m)}},$$

that is a copy of the process of level  $\bar{\gamma}_{m-1}^{(0)}$ , that starts at time  $\mathbf{t}_{(0,m)}$  at the point  $\xi_{(0,m)}$ , conditioned that its lifetime is  $\mathbf{t}_{(0,m-1)} - \mathbf{t}_{(0,m)}$ . From Theorem 6.3 and Lemma 6.1 we get that the whole process defined on  $[0, \mathbf{t}_{(0)}]$  is a copy of  $\mathbf{K}_{\mathbf{t}_{(0)}} \Xi^\xi$ . The intervals of level 1 are, from right to left,  $[\mathbf{t}_{(0,1)}, \mathbf{t}_{(0,0)})$ ,  $[\mathbf{t}_{(0,2)}, \mathbf{t}_{(0,1)})$ ,  $\dots$ ,  $[\mathbf{t}_{(0,m)}, \mathbf{t}_{(0,m-1)})$ ,  $\dots$ . Their left extremes are respectively  $\mathbf{t}_{(0,1)}, \mathbf{t}_{(0,2)}, \dots, \mathbf{t}_{(0,m)}, \dots$  and the points on the boundary associated are  $\xi_{(0,1)}, \xi_{(0,2)}, \dots, \xi_{(0,m)}, \dots$ . We associate to each  $\vec{k} = (0, m)$  the index  $\vec{k}^* = (0, \bar{\gamma}_m^{(0)})$ , then  $\xi_{\vec{k}+1} = \Lambda(\vec{k}^* + 1)$ .

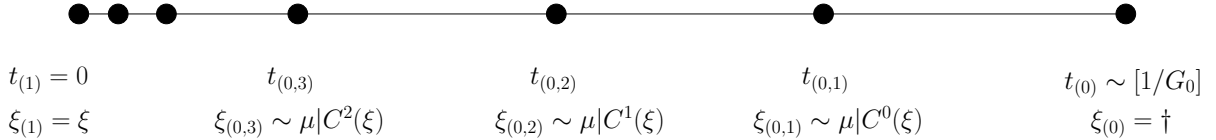


Figure 4: First Step of Simulation

Now we iterate this procedure. We assume the construction has been made up to some  $n \geq 1$ . Consider an interval  $[\mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}}]$  of the level  $n$  characterized by  $\vec{k} = (k_0, \dots, k_n) \in \mathcal{M}$

and its corresponding  $\vec{k}^* \in \mathcal{M}$ . The associated point to its left extreme  $\mathbf{t}_{\vec{k}+1}$  is  $\xi_{\vec{k}+1} = \Lambda(\vec{k}^* + 1)$ . In this interval we need to simulate a copy of the conditional process

$$\mathbf{K}_{\mathbf{t}_{\vec{k}} - \mathbf{t}_{\vec{k}+1}}^{\vec{\gamma}_{k_n}^{\vec{k}^-} \Xi \xi_{\vec{k}+1}}.$$

This requires to simulate exponential random variables distributed as

$$\exp[1/G_{m+1}(\xi_{\vec{k}+1}) - 1/G_m(\xi_{\vec{k}+1})], \quad m > \vec{\gamma}_{k_n}^{\vec{k}^-}.$$

That is, we should consider the variables

$$Z_m^{\vec{k}^*} \quad \text{for } m > \vec{\gamma}_{k_n}^{\vec{k}^-}.$$

We put  $\mathbf{t}_{(\vec{k}, k_n)} = \mathbf{t}_{\vec{k}}$ ,  $\xi_{(\vec{k}, k_n)} = \xi_{\vec{k}}$  and for  $m > k_n$

$$\mathbf{t}_{(\vec{k}, m)} = \mathbf{t}_{\vec{k}+1} + \mathbf{Z}'_m \quad \text{with } \mathbf{Z}' = \mathbf{Z}^{\vec{k}^*}[\vec{\gamma}_{k_n}^{\vec{k}^-}; \mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}}].$$

We also set

$$\underline{\gamma}_m^{\vec{k}} = \underline{\gamma}_m[\mathbf{Z}^{\vec{k}^*}, \vec{\gamma}_{k_n}^{\vec{k}^-}; \mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}}] \quad \text{and} \quad \bar{\gamma}_m^{\vec{k}} = \bar{\gamma}_m[\mathbf{Z}^{\vec{k}^*}, \vec{\gamma}_{k_n}^{\vec{k}^-}; \mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}}].$$

In the interval whose index is  $\vec{h} =: (\vec{k}, m-1)$ , we associate to the left extreme  $\mathbf{t}_{(\vec{k}, m)}$  the point  $\xi_{(\vec{k}, m)} = \Lambda\left((\vec{k}^*, \underline{\gamma}_m^{\vec{k}})\right) = \Lambda\left((\vec{k}^*, \bar{\gamma}_{m-1}^{\vec{k}}) + 1\right)$  that belongs to  $C^{\bar{\gamma}_{m-1}^{\vec{k}}}(\Lambda(\vec{k}^* + 1))$  which was chosen in this set in a uniform way according to  $\mu$ . In this way we define  $\vec{h}^* = (\vec{k}, m-1)^* =: (\vec{k}^*, \bar{\gamma}_{m-1}^{\vec{k}})$  obtaining that

$$\xi_{\vec{h}+1} = \Lambda\left(\vec{h}^* + 1\right).$$

In the interval considered we put a copy of the killed process

$$\mathbf{K}_{\mathbf{t}_{\vec{h}} - \mathbf{t}_{\vec{h}+1}}^{\vec{\gamma}_{m-1}^{\vec{h}^-} \Xi \xi_{\vec{h}+1}}.$$

By construction we have

$$\lim_{m \rightarrow \infty} \downarrow \mathbf{t}_{(\vec{k}, m)} = \mathbf{t}_{\vec{k}+1}$$

Therefore every point  $\mathbf{t}_{\vec{k}}$ ,  $\vec{k} \in \mathcal{M} + 1$ , is an accumulation point of  $(\mathbf{t}_{(\vec{k}, m)})$ . On the other hand for  $m > k_n$  above construction gives

$$\xi_{(\vec{k}, m)} \in C^{\bar{\gamma}_{m-1}^{\vec{k}}} \left( \Lambda(\vec{k}^* + 1) \right).$$

Then  $\lim_{m \rightarrow \infty} \xi_{(\vec{k}, m)} = \Lambda(\vec{k}^* + 1) = \xi_{\vec{k}+1}$ .

In the sequel we adopt the following notation: for  $k \geq 0$  and  $p \geq 1$  by  $k^{[p]}$  we mean the sequence of  $p$  symbols  $k$ , that is  $k^{[p]} = \underbrace{k, \dots, k}_p$ .

**Lemma 6.3** *For every  $\vec{k} = (k_0, \dots, k_n)$  the set of random variables  $(Z_{k_n+1}^{(\vec{k}, k_n^{[p]})} : p \geq 1)$  are independent and identically distributed.*

**Proof.** We must only show they are identically distributed. An inductive argument implies that it suffices to show that  $Z_{k_n+1}^{(\vec{k}, k_n)}$  and  $Z_{k_n+1}^{(\vec{k}, k_n, k_n)}$  have the same distribution.

We have

$$\begin{aligned} Z_{k_n+1}^{(\vec{k}, k_n)} &\sim \exp \left[ \frac{1}{G_{k_n+1}(\Lambda(\vec{k} + 1))} - \frac{1}{G_{k_n}(\Lambda(\vec{k} + 1))} \right], \\ Z_{k_n+1}^{(\vec{k}, k_n, k_n)} &\sim \exp \left[ \frac{1}{G_{k_n+1}(\Lambda((\vec{k}, k_n + 1)))} - \frac{1}{G_{k_n}(\Lambda((\vec{k}, k_n + 1)))} \right]. \end{aligned}$$

We notice that by construction  $\Lambda(\vec{k} + 1) \in C^{k_n}(\Lambda(\vec{k}^- + 1))$ , and  $\Lambda((\vec{k}, k_n + 1)) \in C^{k_n}(\Lambda(\vec{k} + 1)) = C^{k_n}(\Lambda(\vec{k}^- + 1))$ . Since  $G_{k_n+1}, G_{k_n}$  are  $\mathcal{F}_{k_n}$ -measurable we deduce

$$G_{k_n+1}(\Lambda(\vec{k} + 1)) = (G_{k_n+1}(\Lambda((\vec{k}, k_n + 1))))$$

and similarly  $G_{k_n}(\Lambda(\vec{k} + 1)) = (G_{k_n}(\Lambda((\vec{k}, k_n + 1))))$ , proving the result.  $\square$

**Corollary 6.1** *We have*

$$\mathbb{P}\{\forall x > 0, \forall \vec{k} \in \mathcal{M} : \exists p \geq 1, Z_{k_n+1}^{(\vec{k}, k_n^{[p]})} > x \mid \Lambda\} = 1.$$

**Proof.** It is obtained directly from the last Lemma.  $\square$

Therefore we can assume that, conditioned to  $\Lambda$ , for every fixed  $x > 0$  and  $\vec{k} \in \mathcal{M}$ , there exists  $p \geq 1$  such that  $Z_{k_n+1}^{(\vec{k}, k_n^{[p]})} > x$ .

**Corollary 6.2** *In every interval  $[\mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}})$  and for every  $\ell \geq 0$  there exists only a finite number of points  $\mathbf{t}_{\vec{h}}$  in its interior, that is  $\vec{h} = (\vec{k}, k_{n+1}, \dots, k_s)$ , such that  $\underline{\gamma}_{k_s}^{\vec{h}-} = \ell$ .*

**Proof.** Notice that  $T_0 =: \{\mathbf{t}_{\vec{k}} : \underline{\gamma}_{k_n}^{\vec{k}-} = 0\} = \{\mathbf{t}_{(0)}\}$ . The fact that the set  $T_1 =: \{\mathbf{t}_{\vec{k}} : \underline{\gamma}_{k_n}^{\vec{k}-} = 1\}$  is finite follows from the inclusion of events  $\{|T_1| = \infty\} \subseteq \{Z_1^{(0^{[r]})} < t_{(0)} : r \geq 1\}$  and last Corollary. A recurrence argument using Corollary 6.1 finishes the proof.  $\square$

**Theorem 6.4** *The process  $(\Xi_t : t < \Upsilon)$  has a version that is right continuous with left limits and in the set of points  $[0, \Upsilon) \setminus \{\mathbf{t}_{\vec{k}} : \vec{k} \in \mathcal{M}\}$  it is continuous.*



**Proof.** Let us fix an interval  $[\mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}})$ . We denote by  $H = \{\vec{h} : \mathbf{t}_{\vec{h}} \in (\mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}})\}$  and a generic  $\vec{h} \in H$  is denoted by  $\vec{h} = (\vec{k}, k_{n+1}, \dots, k_s)$ . To each  $\ell \geq 0$  we associate the set  $T_{\ell}^{\vec{k}} = \{\mathbf{t}_{\vec{h}} \in (\mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}}) : \gamma_{k_s}^{\vec{h}} = \ell\}$ . Consider the set of nonnegative integers  $L^{\vec{k}} = \{\ell : T_{\ell}^{\vec{k}} \neq \emptyset\}$ , and for each  $\ell \in L^{\vec{k}}$  denote  $\mathbf{t}_{\vec{h}_{\ell}} = \max T_{\ell}^{\vec{k}}$  and put  $\vec{h}_{\ell} = (\vec{k}, k_{n+1}, \dots, k_{s_{\ell}})$ . By construction  $\mathbf{t}_{\vec{h}_{\ell}}$  strictly increases along  $\ell \in L^{\vec{k}}$ .

Assume  $L^{\vec{k}}$  is finite and let  $\ell^*$  be its maximal value. We necessarily have  $\Xi_t = \xi_{\vec{h}_{\ell^*}}$  for  $t \in [\mathbf{t}_{\vec{h}_{\ell^*}}, \mathbf{t}_{\vec{k}})$ , because in the contrary there would be some time  $\tilde{t} \in (\mathbf{t}_{\vec{h}_{\ell^*}}, \mathbf{t}_{\vec{k}})$  for which  $\tilde{t} = {}^r\uparrow$ , contradicting the maximality of  $\mathbf{t}_{\vec{h}_{\ell^*}}$ .

Now assume  $L^{\vec{k}}$  is infinite. Since  $\mathbf{t}_{\vec{h}_{\ell}}$  is increasing, there exists  $t^* = \lim_{\substack{\ell \rightarrow \infty \\ \ell \in L^{\vec{k}}}} \mathbf{t}_{\vec{h}_{\ell}}$ . Observe that for every  $\ell \in L^{\vec{k}}$  and  $t \in (\mathbf{t}_{\vec{h}_{\ell}}, \mathbf{t}_{\vec{k}})$  we have  $\Xi_t \in C^{k_{s_{\ell}}-1}(\Lambda(\vec{h}_{\ell}^-))$ . Then there exists  $\xi^* = \lim_{\substack{\ell \rightarrow \infty \\ \ell \in L^{\vec{k}}}} \xi_{\vec{h}_{\ell}}$ . If  $t^* < \mathbf{t}_{\vec{k}}$ , we can show as before that necessarily  $\Xi_t = \xi^*$  for  $t \in [t^*, \mathbf{t}_{\vec{k}})$ .

Let us summarize. We have shown that at every point  $\{\mathbf{t}_{\vec{h}} : \vec{h} \in H\}$  the killed process is continuous from the right with a limit at the left. Now we take  $t \in (\mathbf{t}_{\vec{k}+1}, \mathbf{t}_{\vec{k}}) \setminus \{\mathbf{t}_{\vec{h}} : \vec{h} \in H\}$ . Assume it is an accumulation point of  $\{\mathbf{t}_{\vec{h}} : \vec{h} \in H\}$ .

If it is not an accumulation point from the right we put  $\vec{h}^*$  the closest element of  $\{\mathbf{t}_{\vec{h}} : \vec{h} \in H\}$  to the right of  $t$ , also let  $\mathbf{t}_{\vec{h}_n}$  be an increasing sequence converging to  $t$ . By the same arguments as before there exists  $\xi^* = \lim_{n \rightarrow \infty} \xi_{\vec{h}_n}$  and we also have  $\Xi_t = \xi^*$  for  $t \in [t, \mathbf{t}_{\vec{h}^*})$ . If it is not an accumulation point from the left we put  $\vec{h}^*$  the closest element of  $\{\mathbf{t}_{\vec{h}} : \vec{h} \in H\}$  to the left of  $t$ . Therefore, by construction, we can assume that the decreasing sequence  $\mathbf{t}_{\vec{h}_n}$  in  $\{\mathbf{t}_{\vec{h}} : \vec{h} \in H\}$  converging to  $t$ , verifies  $\xi_{\vec{h}_n} \in C^{\ell_n}(\xi_{\vec{h}^*})$ , with  $\ell_n$  increasing to  $\infty$  as  $n$  does. Therefore  $\xi_{\vec{h}^*} = \lim_{n \rightarrow \infty} \xi_{\vec{h}_n}$ . Hence  $\Xi_t = \xi^*$  for  $t \in [\mathbf{t}_{\vec{h}^*}, t]$ .

Now assume  $t$  is an accumulation point from the right and the left. Let  $\mathbf{t}_{\vec{h}_n}$  be a decreasing sequence and  $\mathbf{t}_{\vec{l}_n}$  be an increasing sequence, in  $\{\mathbf{t}_{\vec{h}} : \vec{h} \in H\}$ , converging to  $t$ . For  $n$  sufficiently large there exists  $m_n$  and  $\ell_n$ , both converging to  $\infty$  as  $n$  does, such that  $\xi_{\vec{h}_n} \in C^{\ell_n}(\xi_{\vec{l}_{m_n}})$ . Therefore  $\lim_{n \rightarrow \infty} \xi_{\vec{h}_n} = \lim_{n \rightarrow \infty} \xi_{\vec{l}_{m_n}}$  and then  $\xi_t$  is this common limit.

We have shown our construction fulfills the properties stated in the Theorem.  $\square$

**Remark 6.3** We notice that the set of discontinuities for the process  $\Xi$  is given by  $\{{}^n\Upsilon : n \geq 0\} = \{\mathbf{t}_{\vec{k}} : \vec{k} \in \mathcal{M}\}$ .

**Theorem 6.5** If the measure  $\mu$  is atomless then the process  $(\Xi_t : t < \Upsilon)$  has no interval of constancy.

**Proof.** Using the Markov property it is enough to prove that for almost all  $\xi$  and all  $t > 0$  we have

$$\mathbb{P}_{\xi}\{\forall_{0 < s < t} \Xi_s = \xi\} = 0.$$

Since  $\mu$  has no atoms we obtain the existence of a strictly increasing sequence of integers  $(n_i)$ , such that  $C^0(\xi) \not\supseteq C^{n_i}(\xi) \not\supseteq C^{n_{i+1}}(\xi) \downarrow \{\xi\}$ . We consider the random times  ${}^{n_i}\Upsilon$ . At these times the process makes a random selection on  $C^{n_i-1}(\xi)$ , then we have to prove

$$\mathbb{P}_\xi\{{}^{n_i}\Upsilon > t \text{ for all } i\} = 0.$$

We notice that each of these random variables is exponentially distributed with parameter  $1/G_{n_i} \uparrow \infty$  and the result follows.  $\square$

## 6.2 The Markov Process in the Boundary under reflection at the root

The operator  $\underline{\mathbf{W}}^{-1} = W^{-1} - G_0^{-1}\mathbb{E}_\mu = \sum_{n \geq 1} G_n^{-1}(\mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1}))$  generates a (conservative) Markov process. Notice that  $\underline{\mathbf{W}}^{-1}$  has the same form as

$$W^{-1} = \sum_{n \geq 0} G_n^{-1}(\mathbb{E}_\mu(\cdot | \mathcal{F}_n) - \mathbb{E}_\mu(\cdot | \mathcal{F}_{n-1}))$$

where in the last expression  $G_0 \equiv \infty$ . Therefore the analogous of Theorem 6.1 holds.

**Theorem 6.6** *The symmetric kernel*

$$p(t, \xi, \eta) = 1 - e^{-t/G_1(\xi)} + \sum_{n=1}^{\lfloor \xi \wedge \eta \rfloor} \frac{e^{-t/G_n(\xi)} - e^{-t/G_{n+1}(\xi)}}{\mu(C^n(\xi))}, \quad (\xi, \eta) \in \partial_\infty^{reg} \times \partial_\infty^{reg}, \quad t > 0, \quad (6.19)$$

is Markovian (with total mass 1) and the Markov semigroup  $P_t^{\underline{\mathbf{W}}}$  induced in  $L^2(\mu)$  verifies

$$P_t^{\underline{\mathbf{W}}}f = \sum_{n \geq 1} e^{-t/G_n} \left( \mathbb{E}_\mu(f | \mathcal{F}_n) - \mathbb{E}_\mu(f | \mathcal{F}_{n-1}) \right).$$

The infinitesimal generator of this semigroup is an extension of  $-\underline{\mathbf{W}}^{-1}$  defined on  $\mathcal{D}$ .

**Remark 6.4** *The formula (6.19) shares some similarities with the formula (3.1) in [2] (see also (2.9) in [3]) developed for random walks on the  $p$ -adic field. Nevertheless, in our case no homogeneity of the tree is needed.*

Let  $\Xi = (\Xi_t)$  be the Markov (conservative) process associated to the Markov semigroup  $P_t^{\underline{\mathbf{W}}}$ . To simulate the process starting from  $\xi$ , we first generate a sequence of independent identically distributed random variables  $(Y_n : n \geq 1)$  with law  $\exp[1/G_1]$ , and we select a sequence of points  $(\xi_n : n \geq 1)$  independent identically distributed in  $\partial_\infty$  with law  $\mu$ . We define  ${}^1\Upsilon_0 = 0$ ,  $\xi_0 = \xi$ ,  ${}^1\Upsilon_k = Y_1 + \dots + Y_k$ . In each random interval  $[{}^1\Upsilon_k, {}^1\Upsilon_{k+1})$  we put a copy of the process  $\mathbf{K}_{{}^1\Upsilon_{k+1}-{}^1\Upsilon_k} {}^1\Xi^{\xi_k}$ , which is the process  ${}^1\Xi^{\xi_k}$  conditioned to the fact that the killing time  ${}^1\Upsilon$  verifies  ${}^1\Upsilon = {}^1\Upsilon_{k+1} - {}^1\Upsilon_k$ . We summarize the main properties of  $\Xi$  in the following result.

**Theorem 6.7** *The process  $(\Xi_t : t \geq 0)$  has a version that is right continuous with left limits. The set of points of continuity is the complement of  $\{{}^r\Upsilon_k : n \geq 1, k \geq 0\}$ .*

# References

- [1] S. Albeverio, W. Karwowski, Diffusion on the  $p$ -adic numbers. *Gaussian Random Fields (Nagoya 1990)*, World Scientific (1991) 86–99.
- [2] S. Albeverio, W. Karwowski, A random walk on  $p$ -adics the generator and its spectrum. *Stochastic Processes and their Applications* **53** (1994), 1–22.
- [3] S. Albeverio, X. Zhao, Measure-valued branching processes associated with random walks on  $p$ -adics. *Annals of Probability* **28** (2000), 1680–1710.
- [4] S. Albeverio, X. Zhao, On the relation between different constructions of random walks on  $p$ -adics. *Markov Processes and Related Fields* **6** (2000), 239–255.
- [5] D. Aldous, S. Evans, Dirichlet forms on totally disconnected spaces and bipartite Markov chains. *Journal Theoretical Probability* **12** (1999), 839–857.
- [6] W.J. Anderson, *Continuous-time Markov chains: an application-oriented approach*. Springer Series in Statistics, Springer–Verlag (1991).
- [7] C. Berge and A. Ghouila-Houri, *Programmes, jeux et réseaux de transport*. Dunod (1962).
- [8] J.P. Benzécri, *L'Analyse des données*. Dunod (1973).
- [9] R. Blumenthal, R. Gettoor, *Markov Processes and Potential Theory*. Academic Press (1968).
- [10] N. Bouleau, Autour de la variance comme forme de Dirichlet. *Séminaire de Théorie du Potentiel 8, Lecture Notes in Mathematics* **1235** (1989), 39–53.
- [11] P. Cartier, Fonctions harmoniques sur un arbre. *Symposia Mathematica IX*, Academic Press (1972), 203–270.
- [12] D. Cartwright and S. Sawyer, The Martin boundary for general isotropic random walks in a tree. *J. Theoretical Probability* **4** (1991), 111–136.
- [13] K.L. Chung and Z. Zhao, *From Brownian Motion to Schrödinger's equation*. Springer Verlag (1995).
- [14] P. Dartnell, S. Martínez and J. San Martín, Opérateurs filtrés et chaînes de tribus invariantes sur un espace probabilisé dénombrable. *Séminaire de Probabilités XXII Lecture Notes in Mathematics* **1321**, Springer-Verlag (1988).
- [15] C. Dellacherie, S. Martínez and J. San Martín, Ultrametric matrices and induced Markov chains. *Advances in Applied Mathematics* **17** (1996), 169–183.

- [16] C. Dellacherie, S. Martínez and J. San Martín and D. Taïbi. Noyaux potentiels associés à une filtration. *Ann. Inst. Henri Poincaré Prob. et Stat.* **34** (1998), 707–725.
- [17] C. Dellacherie and C. Stricker, Changement de temps et intégrales stochastiques. *Séminaire de Probabilités XI Lectures Notes in Mathematics* **581**, Springer-Verlag (1977).
- [18] B. Derrida, Random-energy model: an exactly solvable model of disordered systems. *Phys. Review B* **24** (1981), 2613-2626.
- [19] S. Evans, Local Properties of Lévy processes on a totally disconnected group. *Journal Theoretical Probability* **2** (1989), 209–259.
- [20] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*. Walter de Gruyter (1994).
- [21] Y. Hu, Potentiel kernels associated with a filtration and forward-backward SDE's. *Potential Analysis* **10** (1998), 103–118.
- [22] B. Hughes, Trees and ultrametric spaces: a categorical equivalence. *Advances in Mathematics* **189** (2004), 148–191.
- [23] R. Lyons with Y. Peres *Probability on Trees and Networks*. <http://mypage.iu.edu/~rdlyons/prbtree/prbtree.html> (2005).
- [24] J. Kemeny, J. Snell, A. Knapp, *Denumerable Markov Chains*. Springer-Verlag, GTM **40**, 2<sup>nd</sup> Edition (1976).
- [25] A. Khrennikov, *p-Adic Valued Distributions in Mathematical Physics*. Kluwer Acad. Publ. (1994).
- [26] A. Kochubei, Hausdorff measure for stable-like process over an infinite extension of a local field. *Journal Theoretical Probability* **15** (2002), 951–972.
- [27] A. Kochubei, Stochastic integrals and stochastic differential equations over the field of  $p$ -adic numbers. *Potential Analysis* **6** (1997), 105–125.
- [28] R. Lyons, Random walks and percolation on trees. *Annals of Probability* **18** (1990), 931-958.
- [29] R. Lyons, Random walks, capacity and percolation on trees. *Annals of Probability* **20** (1992), 2043-2088.
- [30] T. Lyons, A simple criterion for transience of a reversible Markov chain, *Annals of Probability* **11** (1983), 393-402.
- [31] S. Martínez, G. Michon and J. San Martín, Inverses of ultrametric matrices are of Stieltjes types. *SIAM J. Matrix Analysis and its Applications*, **15** (1994), 98–106.

- [32] S. Martínez, D. Remenik and J. San Martín, *Level-wise approximation of a Markov process associated to the boundary of an infinite tree*. Accepted in Journal Theoretical Probability (2006).
- [33] J.J. McDonald, M. Neumann, H. Schneider and M.J. Tsatsomeros, Inverse  $M$ -matrix inequalities and generalized ultrametric matrices. *Linear Algebra and its Applications*, **220** (1995), 321–341.
- [34] M. Mézard, G. Parisi and M. Virasoro, *Spin Glasses and Beyond*. World Scientific (1987).
- [35] R. Nabben and R.S. Varga, A linear algebra proof that the inverse of a strictly ultrametric matrix is a strictly diagonally dominant Stieljes matrix. *SIAM J. Matrix Analysis and its Applications* **15** (1994), 107–113.
- [36] R. Nabben and R.S. Varga, Generalized ultrametric matrices – a class of inverse  $M$ -matrices. *Linear Algebra and its Applications* **220** (1995), 365–390.
- [37] M. Picardello and W. Woess, Martin boundaries of random walks: ends of trees and groups. *Transactions of the AMS* **302** (1987), 185–205.
- [38] R. Rammal, G. Toulouse, M.A. Virasoro, Ultrametricity for physicists. *Rev. Mod. Physics* **58** (1986), 765–788.
- [39] S. Sawyer, Martin Boundaries and Random Walks. *Contemporary Mathematics* **206** (1997), 17–44.